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Newton-like methods for the numerical solution of discrete and continuous two-point boundary value problems.

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Abstract

By using the integral equation formulation of two-point boundary value problems we construct Newton-like methods, where the Jacobian of Newton's method is evaluated only on a subset of the grid points used on the discretization of the problem, for the numerical solution of these problems. If no points are used for the Jacobian we get the method of successive approximations, and if all points are used we get Newton's method. The Newton-like methods constitute a homotopic connection between the method of successive approximations and Newton's method over a set of positive integers. The matrices used in the Newton-like methods are tridiagonal and computation in parallel is possible.

1. INTRODUCTION

We will consider the following boundary value problem

$$(1.1)\quad \begin{cases} Ly = (py')' + qy = f(t,y(t)), & 0 < t < 1, \\ B_1y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2y = b_{21}y(1) + b_{22}y'(1) = d. \end{cases}$$

If $\mu = 0$ is not an eigenvalue of the eigenvalue problem $Ly + \mu y = B_1y = B_2y = 0$ then (1.1) can be formulated as an equivalent integral equation (Stakgold [16])

$$(1.2)\quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \le x \le 1,$$

where

$$(1.3)\quad G(x,t) = \begin{cases} b(x)a(t), & x \ge t, \\ a(x)b(t), & x \le t, \end{cases}$$

is Green's function for L, subject to the boundary conditions B_1 and B_2 of (1.1). Green's function for the boundary value problem (1.1), with $q \neq 0$, is in general not known. However, if we move the term qy to the right hand side, then Green's function for the differential operator of the left hand side will be easy to obtain. The integral equation formulation of two-point boundary value problems has not been a popular tool for the numerical solution of these problems. The reason is that the linearization of (1.2) leads to a linear system with a dense matrix, while the finite-difference approximation of the differential equation in (1.1) will give a tridiagonal matrix. Aulin [6], [10] show how the integal equation (1.2) can be written in a form which give a tridiagonal matrix. We will call this form the determinant formulation of the boundary value problem (or of the integral equation).

For a Fredholm integral equation of the second kind Brakhage [11]

and Atkinson [2], [3] have used an iterative variant of Nyström's method to obtain approximate solutions. Newton's method for the integral equation (1.2) is

$$(1.4)\quad \begin{cases} u(x) - \int_0^1 G(x,t) f_y'(t,y(t))u(t)dt = -\left[y(x) - \int_0^1 G(x,t)f(t,y(t))dt - \right. \\ \left. - \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)}\right], \quad 0 \le x \le 1, \\ y := y + u. \end{cases}$$

Let m,n, and r be positive integers and $n = rm$. Then m and n define two grids, G_m and G_n, on $[0,1]$ with equidistant points. The iterative variant of Nyström's method consist of two steps. First the left hand side of (1.4) is approximated on the coarser grid G_m and the right hand side on G_n. We then obtain a linear system for u on G_m. In the second step intermediate values of u, on G_n, are found by using the discrete version of (1.4) as an interpolation formula. However, this procedure is very ineffective from a numerical point of view. First the left hand side corresponds to a dense $(m+1)\times(m+1)$ matrix, and second the right hand side involves the computation of sums with $n+1$ terms. The number of operations for this method is proportional to n^2. Thus this method cannot compete with the method of finite-difference approximation of (1.1).

The iterative variant of Nyström's method is called a Newton-like method. The following definition has been given by Dennis in [12]:

A <u>Newton-like method</u> is any iterative method of the form

$$(1.5)\quad \begin{cases} x_{n+1} = x_n - A_n^{-1}P(x_n), \quad n = 0,1,\dots \\ x_0 \text{ given in advance,} \end{cases}$$

for generating approximate solutions to $P(x) = 0$, where P is a continuously Fréchet differentiable operator on a Banach space X.

Here $\{A_n\}$ denotes a sequence of invertible linear operators. Obviously A_n should approximate $P'(x_n)$ in some sense.

By using the determinant formulation effective variants of Nyström's method can be obtained. If we apply the determinant formulation to this method, on the grid G_m, we get a system with a tridiagonal matrix. The elements of the vector of the right hand side will now contain sums with $2r$ terms. To compute intermediate values of u we apply the determinant formulation to the iterative variant of Nyström's method on the grid G_n. Then the intermediate values of u are explicitly defined by this Newton-like method.

The advantage of Newton-like methods is that f_y' is evaluated only on the grid G_m and we only have to solve a tridiagonal system with $m+1$ equations. Furthermore the main part of the algorithm is the evaluation of $2m$ sums with $2r$ terms each and these sums can all be evaluated in parallel. These sums are used on both steps of the Newton-like method.

The first step can be interpreted as Newton's method and the second step as the method of successive approximations. Hence the Newton-like method can be considered as a compromise between the method of successive approximations and Newton's method. An optimal choise of m will depend of the amount of work required to compute f_y', the rate of convergence for different valus of m, and the possibility of doing the computations in parallel.

In Aulin [4], [5] Newton-like methods for the numerical solution of the boundary value problem

$$\begin{cases} y'' = f(y(t)), & 0 < t < 1 \\ y(0) = a, \quad y(1) = b, \end{cases}$$

are given. By using the determinant formulation Newton-like methods for the numerical solution of the boundary value problem (1.1) were given in Aulin [6], [9].

In this report we will give a more thorough-going study of Newton-like methods, and as a base for this study we will use Aulin [10]. We will also give Newton-like methods for discrete boundary value problems, i.e. systems of discrete equations with tridiagonal matrices.

2. DETERMINANT FORMULATION OF TWO-POINT BOUNDARY VALUE PROBLEMS

In this chapter we will give a brief outline of the determinant formulation of two-point boundary value problems. For more details see Aulin [10].

Consider the two-point boundary value problem

$$(2.1)\quad \begin{cases} Ly = (py')' + qy = f(t,y(t)), & 0 < t < 1, \\ B_1y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

where $a_{11}, a_{12}, b_{21}, b_{22} \geq 0$, $a_{11} + b_{21} > 0$, $a_{11} + a_{12} > 0$, and $b_{21} + b_{22} > 0$; p', p, q, and f are continuous and $p > 0$ on $[0,1]$. We assume that all eigenvalues μ of the eigenvalue problem

$$(2.2)\quad \begin{cases} Ly + \mu y = 0, \\ B_1y = B_2y = 0, \end{cases}$$

are greater than zero. Then Green's function for L, subject to the boundary conditions B_1 and B_2, exists and is unique (see Stakgold [16] p. 66). To construct Green's function choose $a,b \in C^2$ such that

$$(2.3)\quad \begin{cases} La = 0, \quad B_1a = 0, \quad a \neq 0, \\ Lb = 0, \quad B_2b = 0, \quad b \neq 0. \end{cases}$$

Then Green's function is

$$(2.4)\quad G(x,t) = \begin{cases} b(x)a(t)/C, & x \geq t, \\ a(x)b(t)/C, & x \leq t, \end{cases}$$

where C is the constant $p(ab' - a'b) \neq 0$. In the sequel set $a := a/C$, then the Wronskian of a and b is

$$(2.5)\quad ab' - a'b = 1/p.$$

Now the integral equation formulation of (2.1) is

$$(2.6)\quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \le x \le 1,$$

where $B_2(a) \neq 0$ and $B_1(b) \neq 0$ since all eigenvalues of (2.2) are greater than zero (see Stakgold [16] pp 64-79).

If $a_{12} > 0$ we have

$$(2.7)\quad \frac{1}{B_1(b)} = -\frac{a(0)p(0)}{a_{12}}$$

and if $b_{22} > 0$ we have

$$(2.8)\quad \frac{1}{B_2(a)} = -\frac{b(1)p(1)}{b_{22}}.$$

We note that we make no attempt to derive conditions for the existence and uniqueness of solutions of (2.1), being concerned here only with the numerical analysis problem of determing the condition under which a given numerical method will converge and its rate of convergence. However, it is well known that if the first eigenvalue μ_0 of (2.2) is greater than zero and

$$e \ge f_y'(t,y) \ge \gamma > -\mu_0$$

(e fix constant) for all $(t,y) \in [0,1] \times R$, then (2.1) has a unique solution. In this report we will assume that $\gamma = 0$, i.e.

$$(2.9)\quad 0 \le f_y'(t,y) \le e, \quad (t,y) \in [0,1] \times R, \quad e \ge 0,$$

and of course that $f(t,y)$ is continuously differentiable in y.

Consider the grid $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$. By (2.6) we have

$$y(x) - b(x)\int_0^{x_{i-1}} a(t)f(t,y(t))dt - a(x)\int_{x_{i+1}}^1 b(t)f(t,y(t))dt -$$

$$- \int_{x_{i-1}}^{x_{i+1}} G(x,t)f(t,y(t))dt - \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)} = 0$$

for $i = 1,\ldots,n-1$, $x_{i-1} \le x \le x_{i+1}$. Put

$$X = \int_0^{x_{i-1}} a(t)f(t,y(t))dt + \frac{c}{B_1(b)},$$

$$Y = \int_{x_{i+1}}^1 b(t)f(t,y(t))dt + \frac{d}{B_2(a)},$$

$$C_k = y(x_k) - \int_{x_{i-1}}^{x_{i+1}} G(x_k,t)f(t,y(t))dt, \quad k = i-1,\ i,\ i+1.$$

Then we get

$$\begin{cases} b(x_{i-1})X + a(x_{i-1})Y = G_{i-1} \\ b(x_i)X + a(x_i)Y = G_i \\ b(x_{i+1})X + a(x_{i+1})Y = G_{i+1}. \end{cases}$$

Hence we have an overdetermined linear system with three equations and two unknowns, X and Y. If the integral equation (2.6) has a solution y, then all three equations must be satisfied. This means that the three vectors

$$\begin{bmatrix} b(x_{i-1}) \\ b(x_i) \\ b(x_{i+1}) \end{bmatrix}, \quad \begin{bmatrix} a(x_{i-1}) \\ a(x_i) \\ a(x_{i+1}) \end{bmatrix}, \quad \begin{bmatrix} G_{i-1} \\ G_i \\ G_{i+1} \end{bmatrix},$$

are linearly dependent. Hence we must have

$$(2.10)\qquad \begin{vmatrix} b(x_{i-1}) & a(x_{i-1}) & y(x_{i-1}) - \int_{x_{i-1}}^{x_{i+1}} G(x_{i-1},t)f(t,y(t))dt \\ b(x_i) & a(x_i) & y(x_i) - \int_{x_{i-1}}^{x_{i+1}} G(x_i,t)f(t,y(t))dt \\ b(x_{i+1}) & a(x_{i+1}) & y(x_{i+1}) - \int_{x_{i-1}}^{x_{i+1}} G(x_{i+1},t)f(t,y(t))dt \end{vmatrix} = 0$$

for $i = 1,\dots,n-1$.

Since

$$y(x) - a(x)\int_{x_1}^{1} b(t)f(t,y(t))dt - \int_{1}^{x_1} G(x,t)f(t,y(t))dt -$$

$$- \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)} = 0, \quad 0 \le x \le x_1,$$

we get by the same method

$$(2.11)\qquad \begin{vmatrix} a(0) & y(0) - \int_{0}^{x_1} G(0,t)f(t,y(t))dt - \frac{cb(0)}{B_1(b)} \\ a(x_1) & y(x_1) - \int_{0}^{x_1} G(x_1,t)f(t,y(t))dt - \frac{cb(x_1)}{B_1(b)} \end{vmatrix} = 0$$

and at the other end-point we have

$$(2.12)\qquad \begin{vmatrix} b(x_{n-1}) & y(x_{n-1}) - \int_{x_{n-1}}^{1} G(x_{n-1},t)f(t,y(t))dt - \frac{da(x_{n-1})}{B_2(a)} \\ b(1) & y(1) - \int_{x_{n-1}}^{1} G(1,t)f(t,y(t))dt - \frac{da(1)}{B_2(a)} \end{vmatrix} = 0.$$

We call (2.10), (2.11), and (2.12) the determinant formulation of the boundary value problem (2.1).

Put

$$(2.13)\quad D(t_1,t_2) = \begin{vmatrix} b(t_1) & a(t_1) \\ b(t_2) & a(t_2) \end{vmatrix}, \quad 0 \le t_1 \le t_2 \le 1.$$

In Aulin [10] we have shown that

$$(2.14)\quad D(t_1,t_2) < 0 \quad \text{for} \quad 0 \le t_1 < t_2 \le 1,$$

if the first eigenvalue μ_0 of the eigenvalue problem (2.2) is greater than zero. By determinant manipulation (2.10) can be written as

$$(2.15)\quad \frac{-y(x_{i-1})}{D(x_{i-1},x_i)} + \frac{D(x_{i-1},x_{i+1})}{D(x_{i-1},x_i)D(x_i,x_{i+1})}\,y(x_i) + \frac{-y(x_{i+1})}{D(x_i,x_{i+1})} =$$

$$= \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t,y(t))\,dt, \quad i = 1,\ldots,n-1,$$

where

$$(2.16)\quad \Delta_i(t) = \begin{cases} D(x_{i-1},t)/D(x_{i-1},x_i), & x_{i-1} \le t \le x_i, \\ D(t,x_{i+1})/D(x_i,x_{i+1}), & x_i \le t \le x_{i+1}. \end{cases}$$

If $a_{12} = 0$ then equation (2.11) can be written as $y(0) = c/a_{11}$.
If $a_{12} > 0$ then $a(0) \neq 0$ and equation (2.11) can be written as

$$(2.17)\quad \frac{a(x_1)}{a(0)D(0,x_1)}\,y(0) + \frac{-y(x_1)}{D(0,x_1)} = \int_0^{x_1} \Delta_0(t) f(t,y(t))\,dt + \frac{c}{a(0)B_1(b)},$$

where

$$(2.18)\quad \Delta_0(t) = D(t,x_1)/D(0,x_1), \quad 0 \le t \le x_1.$$

If $b_{22} = 0$ then equation (2.12) can be written as $y(1) = d/b_{21}$.
If $b_{22} > 0$ then $b(1) \neq 0$ and equation (2.12) can be written as

$$(2.19)\quad \frac{-y(x_{n-1})}{D(x_{n-1},1)} + \frac{b(x_{n-1})}{b(1)D(x_{n-1},1)}\,y(1) = \int_{x_{n-1}}^{1} \Delta_n(t) f(t,y(t))\,dt + \frac{d}{b(1)B_2(a)},$$

where

$$(2.20)\quad \Delta_n(t) = D(x_{n-1},t)/D(x_{n-1},1), \qquad x_{n-1} \le t \le 1.$$

In the sequel we will use equations (2.15), (2.17), and (2.19). Moreover, we will make the following interpretation throughout this report: If $a_{12} = 0$ we omit equation (2.17) and substitute the boundary condition $y(0) = c/a_{11}$ into equation (2.15), and if $b_{22} = 0$ we omit equation (2.19) and substitute the boundary condition $y(1) = d/b_{21}$ into (2.15).

We say that a $n \times n$ matrix A is an M-matrix if $A^{-1} \ge 0$ and $a_{i,j} \le 0$, $i \ne j$. A symmetric M-matrix is a Stieltjes matrix.
Let T denote the matrix of the left hand side of the system defined by (2.15), (2.17), and (2.19). In Aulin [10] it is shown that $-T$ is a Stieltjes matrix, and $T^{-1} = G = (g_{i,j})$, where

$$(2.21)\quad g_{i,j} = \begin{cases} b(x_i)a(x_j), & i \ge j, \\ a(x_i)b(x_j), & i \le j, \end{cases}$$

and

$$(2.22)\quad i,j \ge \begin{cases} 0 & \text{if } a_{12} > 0, \\ 1 & \text{if } a_{12} = 0, \end{cases} \qquad i,j \le \begin{cases} n & \text{if } b_{22} > 0, \\ n-1 & \text{if } b_{22} = 0. \end{cases}$$

We will now show a commutative property of the determinant formulation which is very important for the construction of Newton-like methods. Consider (2.15) and put $x_i := x$, $x_{i-1} < x < x_{i+1}$. Then (2.15) can be written as

$$(2.23)\quad y(x) = \frac{D(x,x_{i+1})}{D(x_{i-1},x_{i+1})}\,y(x_{i-1}) + \frac{D(x_{i-1},x)}{D(x_{i-1},x_{i+1})}\,y(x_{i+1}) +$$

$$+ \int_{x_{i-1}}^{x} \frac{D(x,x_{i+1})D(x_{i-1},t)}{D(x_{i-1},x_{i+1})}\,f(t,y(t))dt +$$

$$+ \int_{x}^{x_{i+1}} \frac{D(x_{i-1},x)D(t,x_{i+1})}{D(x_{i-1},x_{i+1})}\,f(t,y(t))dt, \qquad x_{i-1} \le x \le x_{i+1}.$$

Put

$$\begin{cases} a(x) = D(x_{i-1},x)/D(x_{i-1},x_{i+1}), \\ b(x) = D(x,x_{i+1}), \end{cases}$$

for $x_{i-1} \le x \le x_{i+1}$. Then (2.23) defines an integral equation for y in the interval $[x_{i-1},x_{i+1}]$ and this equation is of the same type as (2.6). Thus the determinant formulation of (2.23) at the points $x_{i-1} \le \xi_1 < \xi_2 < \xi_3 \le x_{i+1}$ is given by (2.15). With

$$(2.24)\quad T(x,t) = \begin{vmatrix} D(x,x_{i+1}) & D(x_{i-1},x)/D(x_{i-1},x_{i+1}) \\ D(t,x_{i+1}) & D(x_{i-1},t)/D(x_{i-1},x_{i+1}) \end{vmatrix}$$

we can write

$$(2.25)\quad \frac{-y(\xi_1)}{T(\xi_1,\xi_2)} - \frac{T(\xi_1,\xi_3)}{T(\xi_1,\xi_2)T(\xi_2,\xi_3)}\,y(\xi_2) + \frac{-y(\xi_3)}{T(\xi_2,\xi_3)} =$$

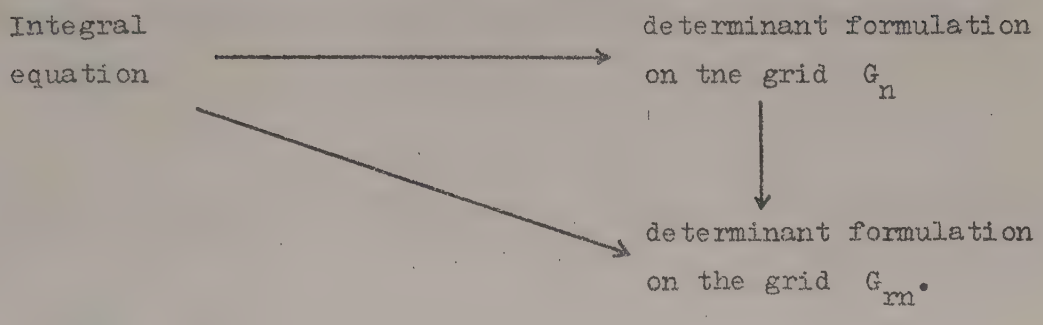
$$= \int_{\xi_1}^{\xi_2} \frac{T(\xi_1,t)}{T(\xi_1,\xi_2)}\,f(t,y(t))dt + \int_{\xi_2}^{\xi_3} \frac{T(t,\xi_3)}{T(\xi_2,\xi_3)}\,f(t,y(t))dt.$$

But it is easy to see that $T(x,t) = D(x,t)$ which means that equation (2.25) is the same equation as (2.15) with $x_{i-1} = \xi_1$, $x_i = \xi_2$, $x_{i+1} = \xi_3$.

Let $G_n = \{ x_i \mid 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1 \}$ be a subset of

$$G_{rn} = \bigcup_{i=0}^{n-1} \{ x_{i,j} \mid x_i = x_{i,0} < x_{i,1} < \cdots < x_{i,r-1} < x_{i,r} = x_{i+1} \}.$$

Then we have the following commutative diagram



In Aulin [10] it is shown that the determinant formulation defined by (2.15), (2.17), and (2.19) exists, even if the first eigenvalue μ_0 of the eigenvalue problem (2.2) is zero (i.e. when Green's function doesn't exist), if the grid space of G_n is sufficiently small. However we will only consider the case $\mu_0 > 0$ in this report.

3. DERIVATION OF NEWTON-LIKE METHODS FOR CONTINUOUS BOUNDARY VALUE PROBLEMS

We will define Newton-like methods for the numerical solution of the boundary value problem (2.1). Newton's method for the integral equation (2.6) is

$$(3.1)\quad \begin{cases} u(x) - \int_0^1 G(x,t) f_y'(t,y(t))u(t)dt = -\left[y(x) - \int_0^1 G(x,t)f(t,y(t))dt - \right. \\ \left. - \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)}\right], \\ y(x) := y(x) + u(x), \end{cases}$$

for $0 \le x \le 1$.

Consider the grid

$$G_n:\ 0 = x_0 < x_1 < \dots \quad x_{n-1} < x_n = 1,$$

and the subgrid $(m \le n)$

$$G_m:\ 0 = x_{n_0} < x_{n_1} < \dots < x_{n_m} = 1,$$

where n_i are integers and $0 \le n_i \le n$, $i = 0,\dots,m$. For simplicity put $t_i = x_{n_i}$, $i = 0,\dots,m$.

For a Fredholm integral equation of the second kind Brakhage [11] and Atkinson [2], [3] have used an iterative variant of Nyström's method to obtain approximate solutions. We get this method if we approximate the left hand side of the integral equation in (3.1) on the grid G_m and the right hand side on the grid G_n. We will first consider the trapezoidal rule. Thus put

$$(3.2)\quad \omega_j = \begin{cases} t_1/2, & j = 0, \\ (t_{j+1} - t_{j-1})/2, & j = 1,\dots,m-1, \\ (1 - t_{m-1})/2, & j = m, \end{cases}$$

and

$$(3.3)\qquad w_k = \begin{cases} x_1/2, & k = 0, \\ (x_{k+1} - x_{k-1})/2, & k = 1,\dots,n-1, \\ (1 - x_{n-1})/2, & k = n. \end{cases}$$

Then we get the following Newton-like method

$$(3.4)\qquad \begin{cases} u(x) - \sum_{j=0}^{m} {}_jG(x,t_j)f'_y(t_j,y(t_j))u(t_j) = \\ \\ = -\left[y(x) - \sum_{k=0}^{n} w_kG(x,x_k)f(x_k,y(x_k)) - \dfrac{cb(x)}{B_1(b)} - \dfrac{da(x)}{B_2(a)}\right], \\ \\ y(x) := y(x) + u(x), \quad 0 \le x \le 1, \end{cases}$$

for generating approximate solutions to the sum equation

$$y(x) = \sum_{k=0}^{n} w_kG(x,x_k)f(x_k,y(x_k)) + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \le x \le 1.$$

For simplicity we have used the same symbols for the unknowns u and y in both (3.1) and (3.4), and we will apply the same principle throughout this report.

For $x := t_i$, $i = 0,\dots,m$, the sum equation (3.4) defines a linear system for the unknowns $u(t_i)$, $i = 0,\dots,m$. If the matrix that corresponds to this system is nonsingular we can solve the equations for $u(t_i)$, $i = 0,\dots,m$. After that we can use the sum equation (3.4) as an interpolation formula to compute intermediate values $u(x_i)$, $i = 0,\dots,n$. This procedure is called a Newton-like method. However, this procedure is very ineffective from the numerical point of view. First, the left hand side of the sum equation in (3.4) corresponds to a dense $(m+1)\times(m+1)$ matrix, and second, the right hand side involves the computation of sums with $n+1$ terms.

To get a form better adapted to numerical computations we apply the

determinant formulation to the sum equation in (3.4) on the grid G_m. By exactly the same method as used for integral equations we get

$$(3.5)\quad \frac{-u(t_{i-1})}{D(t_{i-1},t_i)} + \frac{D(t_{i-1},t_{i+1})}{D(t_{i-1},t_i)D(t_i,t_{i+1})}u(t_i) + \frac{-u(t_{i+1})}{D(t_i,t_{i+1})} -$$

$$- \omega_i f'_y(t_i,y(t_i))u(t_i) =$$

$$= -\left\{\frac{-y(t_{i-1})}{D(t_{i-1},t_i)} + \frac{D(t_{i-1},t_{i+1})}{D(t_{i-1},t_i)D(t_i,t_{i+1})}y(t_i) + \frac{-y(t_{i+1})}{D(t_i,t_{i+1})} -\right.$$

$$\left. - \sum_{k=n_{i-1}+1}^{n_{i+1}-1} w_k \Delta_i(x_k) f(x_k,y(x_k))\right\},$$

for $i = 1,\ldots,m-1$, where

$$(3.6)\quad \Delta_i(x) = \begin{cases} D(t_{i-1},x)/D(t_{i-1},t_i), & t_{i-1} \le x \le t_i \\ D(x,t_{i+1})/D(t_i,t_{i+1}), & t_i \le x \le t_{i+1}. \end{cases}$$

If $a_{12} > 0$ we get

$$(3.7)\quad \frac{a(t_1)}{a(0)D(0,t_1)}u(0) + \frac{-u(t_1)}{D(0,t_1)} - \omega_0 f'_y(0,y(0))u(0) =$$

$$= -\left\{\frac{a(t_1)}{a(0)D(0,t_1)}y(0) + \frac{-y(t_1)}{D(0,t_1)} - \sum_{k=0}^{n_1-1} w_k \Delta_0(x_k) f(x_k,y(x_k)) -\right.$$

$$\left. - \frac{c}{a(0)B_1(b)}\right\},$$

where

$$(3.8)\quad \Delta_0(x) = D(x,t_1)/D(0,t_1), \quad 0 \le x \le t_1.$$

If $b_{22} > 0$ we get

$$(3.9)\quad \frac{-u(t_{m-1})}{D(t_{m-1},1)} + \frac{b(t_{m-1})}{b(1)D(t_{m-1},1)}\,u(1) - w_m f'_y(1,y(1))u(1) =$$

$$= -\left\{\frac{-y(t_{m-1})}{D(t_{m-1},1)} + \frac{b(t_{m-1})}{b(1)D(t_{m-1},1)}\,y(1) -\right.$$

$$\left. - \sum_{k=n_{m-1}+1}^{n} w_k \Delta_k(x_k) f(x_k,y(x_k)) - \frac{d}{b(1)B_2(a)}\right\},$$

where

$$(3.10)\quad \Delta_m(x) = D(t_{m-1},x)/D(t_{m-1},1), \qquad t_{m-1} \le x \le 1.$$

Now the dense matrix of the sum equation in (3.4) has been reduced to a tridiagonal matrix of the system defined by (3.5), (3.7), and (3.9).

To compute intermediate values of u consider the determinant formulation of equation (3.4) at the points t_i, x_j, t_{i+1} where $t_i < x_j < t_{i+1}$ $(n_i < j < n_{i+1})$. We get

$$(3.11)\quad \frac{-u(t_i)}{D(t_i,x_j)} + \frac{D(t_i,t_{i+1})}{D(t_i,x_j)D(x_j,t_{i+1})}\,u(x_j) + \frac{-u(t_{i+1})}{D(x_j,t_{i+1})} =$$

$$= -\left\{\frac{-y(t_i)}{D(t_i,x_j)} + \frac{D(t_i,t_{i+1})}{D(t_i,x_j)D(x_j,t_{i+1})}\,y(x_j) + \frac{-y(t_{i+1})}{D(x_j,t_{i+1})} -\right.$$

$$- \sum_{k=n_i+1}^{j} w_k \frac{D(t_i,x_k)}{D(t_i,t_{i+1})} f(x_k,y(x_k)) -$$

$$\left. - \sum_{k=j+1}^{n_{i+1}} w_k \frac{D(x_k,t_{i+1})}{D(t_i,t_{i+1})} f(x_k,y(x_k))\right\},$$

for $j = n_i + 1,\ldots, n_{i+1} - 1$, $i = 0,1,\ldots,m-1$.

Now (3.11) can be written as

$$(3.12)\quad y(x_j) + u(x_j) =$$

$$= \frac{D(x_j,t_{i+1})}{D(t_i,t_{i+1})}\left[y(t_i) + u(t_i)\right] + \frac{D(t_i,x_j)}{D(t_i,t_{i+1})}\left[y(t_{i+1}) + u(t_{i+1})\right] +$$

$$+ \frac{D(x_j,t_{i+1})}{D(t_i,t_{i+1})} \sum_{k=n_i+1}^{j} w_k D(t_i,x_k) f(x_k,y(x_k)) +$$

$$+ \frac{D(t_i,x_j)}{D(t_i,t_{i+1})} \sum_{k=j+1}^{n_i+1} w_k D(x_k,t_{i+1}) f(x_k,y(x_k)),$$

for $j = n_i+1,\ldots, n_{i+1}-1$, $i = 0,\ldots,m-1$.

We note that $y(x_j) + u(x_j)$ is explicitly defined by (3.12) once the system (3.5), (3.7), (3.9) has been solved. Moreover the large sums in (3.4) have been splitted up into smaller sums.

A new y follows from

$$y(x_j) := y(x_j) + u(x_j), \quad j = 0,\ldots,n.$$

We say that the linear system defined by (3.5), (3.7), and (3.9) followed by equation (3.12) define a Newton-like method for the sum equation

$$(3.13')\quad y(x_j) = \sum_{k=0}^{n} w_k G(x_j,x_k) f(x_k,y(x_k)) + \frac{cb(x_j)}{B_1(b)} + \frac{da(x_j)}{B_2(a)},$$

for $j = 0,1,\ldots,n$, or equivalent, for the determinant formulation of (3.13') which is defined by the following system

$$
(3.13'')\quad
\begin{cases}
\dfrac{a(x_1)}{a(0)D(0,x_1)}\,y(0) + \dfrac{-\,y(x_1)}{D(0,x_1)} = w_0 f(0,y(0)) + \dfrac{c}{a(0)B_1(b)}, \\
\text{if } a_{12} > 0, \\
\dfrac{-\,y(x_{j-1})}{D(x_{j-1},x_j)} + \dfrac{D(x_{j-1},x_{j+1})}{D(x_{j-1},x_j)D(x_j,x_{j+1})}\,y(x_j) + \dfrac{-\,y(x_{j+1})}{D(x_j,x_{j+1})} = \\
= w_j f(x_j,y(x_j)), \quad j = 1,\ldots,n-1, \\
\dfrac{-\,y(x_{n-1})}{D(x_{n-1},1)} + \dfrac{b(x_{n-1})}{b(1)D(x_{n-1},1)}\,y(1) = w_n f(1,y(1)) + \dfrac{d}{b(1)B_1(a)}, \\
\text{if } b_{22} > 0.
\end{cases}
$$

This system defines an $O(h^2)$ approximation of the boundary value problem (2.1) (see Aulin [10]). By (2.15), (2.17), (2.19), and (3.13'') we note that the determinant formulation of the integral equation (2.6) commute with the trapezoidal rule. Thus we also obtain the system (3.5), (3.7), (3.9) if we first form the determinant formulation of (3.1) on the grid G_m, and then approximate the left hand sides of these equations by the trapezoidal rule on the grid G_m and the right hand sides by the same rule but on the finer grid G_n. Moreover if we approximate the determinant formulation of (3.1) at the points t_i, x_j, t_{i+1} by the same method as above, we get (3.11).

For $m = 0$ let $G_0 = \emptyset$. Then (3.4) defines the method of successive approximations for the sum equation (3.13'). If $m = n$ then (3.4) defines Newton's method for (3.13'') (or equivalently (3.13')). Further more we can consider the system (3.5), (3.7), (3.9) as a step in Newton's method and equation (3.12) as a step in the method of successive approximations. Hence the Newton-like methods

defined by (3.5), (3.7), (3.9), and (3.12) can be interpreted as a homotopic connection between the method of successive approximations and Newton's method over a set of positive integers.

We will now give a brief outline of the algorithm for Newton-like methods with $0 < m < n$. This algorithm is almost the same as the algorithm for Newton's method. The only difference is the evaluation of the sums in (3.5), (3.7), and (3.9). In Newton's method these sums consist of one term. The sums in (3.5), (3.7), and (3.9) are evaluated by the following algorithm

$$(3.14)\begin{cases} SU(i,0) = 0; \\ SU(i,k+1) = SU(i,k) + w_{n_i+k+1} D(t_i, x_{n_i+k+1}) f(x_{n_i+k+1}, y(x_{n_i+k+1})); \\ k = 0,1,\ldots,n_{i+1} - n_i - 1, \end{cases}$$

$$(3.15)\begin{cases} SD(i,n_{i+1}-n_i) = 0; \\ SD(i,n_{i+1} - n_i - k - 1) = SD(i,n_{i+1} - n_i - k) + \\ + w_{n_{i+1}-k-1} D(x_{n_i-k-1}, t_{i+1}) f(x_{n_i-k-1}, y(x_{n_{i+1}-k-1})); \\ k = 0,1,\ldots,n_{i+1} - n_i - 1, \end{cases}$$

for $i = 0,\ldots,m-1$. Thus we have

$$(3.16)\quad \sum_{k=n_{i-1}+1}^{n_{i+1}-1} w_k \Delta_i(x_k) f(x_k, y(x_k)) =$$

$$= SU(i-1, n_i - n_{i-1})/D(t_{i-1}, t_i) + SD(i,1)/D(t_i, t_{i+1}),$$

for $i = 1,\ldots,m-1$,

$$(3.17)\quad \sum_{k=0}^{n_1-1} w_k \Delta_0(x_k) f(x_k, y(x_k)) = SD(0,0)/D(0, t_1),$$

and

$$(3.18)\qquad \sum_{k=n_{m-1}+1}^{n} w_k \Delta_m(x_k) f(x_k, y(x_k)) = SU(m-1, n-n_{m-1})/D(t_{m-1}, 1).$$

By (3.14) and (3.15) the sums in (3.12) can be written as

$$(3.19)\qquad \frac{D(x_j, t_{i+1})}{D(t_i, t_{i+1})} \sum_{k=n_i+1}^{j} w_k D(t_i, x_k) f(x_k, y(x_k)) +$$

$$+ \frac{D(t_i, x_j)}{D(t_i, t_{i+1})} \sum_{k=j+1}^{n_{i+1}-1} w_k D(x_k, t_{i+1}) f(x_k, y(x_k)) =$$

$$= \left[D(x_j, t_{i+1}) SU(i, j-n_i) + D(t_i, x_j) SD(i, j+1-n_i) \right] / D(t_i, t_{i+1}),$$

for $j = n_i+1, \ldots, n_{i+1}-1$, $i = 0, \ldots, m-1$.

We now make the following important observation: The $2m$ vectors $SU(i,\cdot)$ and $SD(i,\cdot)$ can all be computed in parallel.

The advantage of Newton-like methods is that f_y' only is evaluated on the grid G_m and that we only have to solve a tridiagonal system with $m+1$ equations. For relative small m the main part of the algorithm is the evaluation of $2m$ sums and these sums can all be evaluated in parallel. An optimal choise of m will depend on the amount of work required to compute f_y', the rate of convergence for different values of m, and the possibility of doing the computations in parallel. Newton-like methods can also be used when the system of Newton's method is too large to be solved directly, mostly because of computer memory size limitations.

The discretization (3.13'') is an $O(h^2)$ approximation of the boundary value problem (2.1). We will now give Newton-like methods for an $O(h^4)$ discretization of the determinant formulation of the boundary value problem (2.1) on the grid $G_n = \left\{ \frac{i}{n} \mid i = 0, \ldots, n \right\}$.

By using the trapezoidal rule with "end correction" we get the following $O(h^4)$ discretization of (2.15)

$$(3.20)\quad \frac{-y(\frac{i-1}{n})}{D(\frac{i-1}{n},\frac{i}{n})} + \frac{D(\frac{i-1}{n},\frac{i+1}{n})}{D(\frac{i-1}{n},\frac{i}{n})D(\frac{i}{n},\frac{i+1}{n})}\, y(\frac{i}{n}) + \frac{-y(\frac{i+1}{n})}{D(\frac{i}{n},\frac{i+1}{n})} = hf(\frac{i}{n},y(\frac{i}{n})) +$$

$$+ \frac{h^2}{12}\Big[\frac{-f(\frac{i-1}{n},y(\frac{i-1}{n}))}{D(\frac{i-1}{n},\frac{i}{n})p(\frac{i-1}{n})} + \frac{D(\frac{i-1}{n},\frac{i+1}{n})}{D(\frac{i-1}{n},\frac{i}{n})D(\frac{i}{n},\frac{i+1}{n})}\,\frac{f(\frac{i}{n},y(\frac{i}{n}))}{p(\frac{i}{n})} +$$

$$+ \frac{-f(\frac{i+1}{n},y(\frac{i+1}{n}))}{D(\frac{i}{n},\frac{i+1}{n})p(\frac{i+1}{n})}\Big],$$

for $i = 1,\ldots,n-1$, $h = 1/n$.

For equation (2.17) the trapezoidal rule with "end correction" is not a suitable approximation for the integral . The reason is that the discrete equation will contain a derivative of f at the point $t = 0$. Instead we use Simpson's rule to approximate the integral in equation (2.17). However we will consider the determinant formulation at the grid points 0 and $\frac{2}{n}$ instead of 0 and $\frac{1}{n}$. Thus if $a_{12} > 0$ we get

$$(3.21)\quad \frac{a(\frac{2}{n})}{a(0)D(0,\frac{2}{n})}\, y(0) + \frac{-y(\frac{2}{n})}{D(0,\frac{2}{n})} = \frac{h}{3}\Big[f(0,y(0)) + 4\,\frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})}\, f(\frac{1}{n},y(\frac{1}{n}))\Big] +$$

$$+ \frac{c}{a(0)B_1(b)}.$$

By (3.20) we can eliminate $y(\frac{2}{n})$ in (3.21), we get

$$(3.22)\quad \frac{a(\frac{1}{n})}{a(0)D(0,\frac{1}{n})}\,y(0) + \frac{-\,y(\frac{1}{n})}{D(0,\frac{1}{n})} = \frac{h}{3}\left[f(0,y(0)) + \frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})}\,f(\frac{1}{n},y(\frac{1}{n}))\right] -$$

$$- \frac{h^2}{12}\,\frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})}\left[\frac{-f(0,y(0))}{D(0,\frac{1}{n})p(0)} + \frac{D(0,\frac{2}{n})f(\frac{1}{n},y(\frac{1}{n}))}{D(0,\frac{1}{n})D(\frac{1}{n},\frac{2}{n})p(\frac{1}{n})} + \frac{-f(\frac{2}{n},y(\frac{2}{n}))}{D(\frac{1}{n},\frac{2}{n})p(\frac{2}{n})}\right] +$$

$$+ \frac{c}{a(0)B_1(b)}\,.$$

If $b_{22} > 0$ we get, by the same method,

$$(3.23)\quad \frac{-y(\frac{n-1}{n})}{D(\frac{n-1}{n},1)} + \frac{b(\frac{n-1}{n})}{b(1)D(\frac{n-1}{n},1)}\,y(1) = \frac{h}{3}\left[\frac{D(\frac{n-2}{n},\frac{n-1}{n})}{D(\frac{n-2}{n},1)}\,f(\frac{n-1}{n},y(\frac{n-1}{n})) +\right.$$

$$\left. + f(1,y(1))\right] - \frac{h^2}{12}\,\frac{D(\frac{n-2}{n},\frac{n-1}{n})}{D(\frac{n-2}{n},1)}\left[\frac{-f(\frac{n-2}{n},y(\frac{n-2}{n}))}{D(\frac{n-2}{n},\frac{n-1}{n})p(\frac{n-2}{n})} +\right.$$

$$\left. + \frac{D(\frac{n-2}{n},1)f(\frac{n-1}{n},y(\frac{n-1}{n}))}{D(\frac{n-2}{n},\frac{n-1}{n})D(\frac{n-1}{n},1)p(\frac{n-1}{n})} + \frac{-\,f(1,y(1))}{D(\frac{n-1}{n},1)p(1)}\right] + \frac{d}{b(1)B_2(a)}\,.$$

Equations (3.20), (3.22), and (3.23) define an $O(h^4)$ approximation of the boundary value problem (2.1) (see Aulin [10]).

The integral equation (2.6) can be written as

$$(3.24)\quad y(\tfrac{i}{n}) = b(\tfrac{i}{n})a(0)\int_0^{\frac{2}{n}} \frac{D(t,\frac{2}{n})}{D(0,\frac{2}{n})}\,f(t,y(t))dt +$$

$$+ a(\tfrac{i}{n})b(1)\int_{\frac{n-2}{n}}^{1} \frac{D(\frac{n-2}{n},t)}{D(\frac{n-2}{n},1)}\,f(t,y(t))dt +$$

$$+\left\{\int_0^1 G(\tfrac{i}{n},t)f(t,y(t))dt - b(\tfrac{i}{n})a(0)\int_0^{\frac{2}{n}} \frac{D(t,\frac{2}{n})}{D(0,\frac{2}{n})}\,f(t,y(t))dt -\right.$$

$$- a(\tfrac{i}{n})b(1) \int_{\frac{n-2}{n}}^{1} \frac{D(\frac{n-2}{n},t)}{D(\frac{n-2}{n},1)} f(t,y(t))dt \Big\} + \frac{b(\tfrac{i}{n})c}{B_1(b)} + \frac{a(\tfrac{i}{n})d}{B_2(a)},$$

for $i = 0,\ldots,n$. If we approximate the first two integrals by Simpson's rule and the other integrals by the trapezoidal rule with "end correction" we get the sum equation formulation of (3.20), (3.22), and (3.23).

We will now give Newton-like methods for the discrete determinant formulation (3.20), (3.22), and (3.23). Let m,n, and r be positive integers and $n = rm$. Then m and n define two grids, $G_m = \{\frac{i}{m} \quad i = 0,\ldots,m\}$ and $G_n = \{\frac{k}{n} \quad k = 0,\ldots,n\}$, on $[0,1]$ with equidistant points. On the grid G_m there are two quadrature rules which can be used, the trapezoidal rule or the trapezoidal rule with "end correction". We will first use the trapezoidal rule with "end correction", and we assume that $m \geq 2$. Consider Newton's method for (3.24). If we approximate the left hand side on the grid G_m and the right hand side on the grid G_n with the trapezoidal rule with "end correction" and apply the determinant formulation to this sum equation on the grid G_m, we get

$$(3.25)\quad \Big[\frac{a(\frac{1}{m})}{a(0)D(0,\frac{1}{m})} - (\frac{1}{3m} + \frac{1}{12m^2}\frac{D(\frac{1}{m},\frac{2}{m})}{D(0,\frac{2}{m})D(0,\frac{1}{m})p(\frac{1}{m})})f_y'(0,y(0))\Big]u(0) -$$

$$-\Big[\frac{1}{D(0,\frac{1}{m})} + (\frac{1}{3m}\frac{D(\frac{1}{m},\frac{2}{m})}{D(0,\frac{2}{m})} - \frac{1}{12m^2}\frac{1}{D(0,\frac{1}{m})p(\frac{1}{m})})f_y'(\tfrac{1}{m},y(\tfrac{1}{m}))\Big]u(\tfrac{1}{m}) -$$

$$-\frac{1}{12m^2}\frac{f_y'(\frac{2}{m},y(\frac{2}{m}))}{D(0,\frac{2}{m})p(\frac{2}{m})}u(\tfrac{2}{m}) =$$

$$= -\Big\{\frac{a(\frac{1}{m})}{a(0)D(0,\frac{1}{m})}y(0) - \frac{y(\frac{1}{m})}{D(0,\frac{1}{m})} - \frac{1}{3n}\Big[f(0,y(0)) + \frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})}f(\tfrac{1}{n},y(\tfrac{1}{n}))\Big] -$$

$$- \frac{1}{n} \sum_{k=1}^{r-1} \Delta_0(\tfrac{k}{n}) f(\tfrac{k}{n}, y(\tfrac{k}{n})) - \frac{1}{12n^2} \frac{1}{D(0,\frac{1}{m}) D(0,\frac{2}{n})} \Big[- D(\tfrac{2}{n},\tfrac{1}{m}) \frac{f(0,y(0))}{p(0)}$$

$$+ D(0,\tfrac{1}{m}) \frac{f(\frac{2}{n}, y(\frac{2}{n}))}{p(\frac{2}{n})} - D(0,\tfrac{2}{n}) \frac{f(\frac{1}{m}, y(\frac{1}{m}))}{p(\frac{1}{m})} \Big] - \frac{c}{a(0) B_1(b)} \Big\}, \text{ if } a_{12} > 0,$$

$$(3.26) \quad \frac{-1}{D(\frac{i-1}{m},\frac{i}{m})} \Big[1 - \frac{1}{12m^2} \frac{f'_y(\frac{i-1}{m}, y(\frac{i-1}{m}))}{p(\frac{i-1}{m})} \Big] u(\tfrac{i-1}{m}) + \Big[\frac{D(\frac{i-1}{m},\frac{i+1}{m})}{D(\frac{i-1}{m},\frac{i}{m}) D(\frac{i}{m},\frac{i+1}{m})} -$$

$$- (\frac{1}{m} + \frac{1}{12m^2} \frac{D(\frac{i-1}{m},\frac{i+1}{m})}{D(\frac{i-1}{m},\frac{i}{m}) D(\frac{i}{m},\frac{i+1}{m}) p(\frac{i}{m})}) f'_y(\tfrac{i}{m}, y(\tfrac{i}{m})) \Big] u(\tfrac{i}{m}) -$$

$$- \frac{1}{D(\frac{i}{m},\frac{i+1}{m})} \Big[1 - \frac{1}{12m^2} \frac{f'_y(\frac{i+1}{m}, y(\frac{i+1}{m}))}{p(\frac{i+1}{m})} \Big] u(\tfrac{i+1}{m}) =$$

$$= - \Big\{ \frac{- y(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \frac{D(\frac{i-1}{m},\frac{i+1}{m})}{D(\frac{i-1}{m},\frac{i}{m}) D(\frac{i}{m},\frac{i+1}{m})} y(\tfrac{i}{m}) + \frac{- y(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -$$

$$- \frac{1}{n} \sum_{k=r(i-1)+1}^{r(i+1)-1} \Delta_i(\tfrac{k}{n}) f(\tfrac{k}{n}, y(\tfrac{k}{n})) - \frac{1}{12n^2} \Big[\frac{-f(\frac{i-1}{m}, y(\frac{i-1}{m}))}{D(\frac{i-1}{m},\frac{i}{m}) p(\frac{i-1}{m})} +$$

$$+ \frac{D(\frac{i-1}{m},\frac{i+1}{m}) f(\frac{i}{m}, y(\frac{i}{m}))}{D(\frac{i-1}{m},\frac{i}{m}) D(\frac{i}{m},\frac{i+1}{m}) p(\frac{i}{m})} + \frac{-f(\frac{i+1}{m}, y(\frac{i+1}{m}))}{D(\frac{i}{m},\frac{i+1}{m}) p(\frac{i+1}{m})} \Big] \Big\}, \quad i = 1, \ldots, m-1,$$

and

$$(3.27) \quad - \frac{1}{12m^2} \frac{f'_y(\frac{m-2}{m}, y(\frac{m-2}{m}))}{D(\frac{m-2}{m}, 1) p(\frac{m-2}{m})} u(\tfrac{m-2}{m}) - \Big[\frac{1}{D(\frac{m-1}{m}, 1)} +$$

$$+ (\frac{1}{3m} \frac{D(\frac{m-2}{m},\frac{m-1}{m})}{D(\frac{m-2}{m}, 1)} - \frac{1}{12m^2} \frac{1}{D(\frac{m-1}{m}, 1) p(\frac{m-1}{m})} f'_y(\tfrac{m-1}{m}, y(\tfrac{m-1}{m})) \Big] u(\tfrac{m-1}{m}) +$$

$$+ \Big[\frac{b(\frac{m-1}{m})}{b(1) D(\frac{m-1}{m}, 1)} - (\frac{1}{3m} + \frac{1}{12m^2} \frac{D(\frac{m-2}{m},\frac{m-1}{m})}{D(\frac{m-2}{m}, 1) D(\frac{m-1}{m}, 1) p(1)}) f'_y(1, y(1)) \Big] u(1) =$$

$$= -\Bigg\{ \frac{b(\frac{m-1}{m})}{b(1)D(\frac{m-1}{m},1)}\, y(1) + \frac{-\,y(\frac{m-1}{m})}{D(\frac{m-1}{m},1)} - \frac{1}{3n}\Big[f(1,y(1)) +$$

$$+ \frac{D(\frac{n-2}{2},\frac{n-1}{n})}{D(\frac{n-2}{n},1)}\, f(\tfrac{n-1}{n},y(\tfrac{n-1}{n}))\Big] - \frac{1}{n}\sum_{k=n-r+1}^{n-1} \Delta_m(\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) -$$

$$- \frac{1}{12n^2}\,\frac{1}{D(\frac{m-1}{m},1)D(\frac{n-2}{n},1)}\Big[- D(\tfrac{n-2}{n},1)\,\frac{f(\frac{m-1}{m},y(\frac{m-1}{m}))}{p(\frac{m-1}{m})} +$$

$$+ D(\tfrac{m-1}{m},1)\,\frac{f(\frac{n-2}{n},y(\frac{n-2}{n}))}{p(\frac{n-2}{n})} - D(\tfrac{m-1}{m},\tfrac{n-2}{n})\,\frac{f(1,y(1))}{p(1)}\Big] -$$

$$- \frac{d}{b(1)B_2(a)}\Bigg\}, \quad \text{if} \quad b_{22} > 0,$$

where

$$\begin{cases} \Delta_0(x) = \dfrac{D(x,\frac{1}{m})}{D(0,\frac{1}{m})}, & 0 \le x \le \frac{1}{m}, \\[2ex] \Delta_i(x) = \begin{cases} \dfrac{D(\frac{i-1}{m},x)}{D(\frac{i-1}{m},\frac{i}{m})}, & \frac{i-1}{m} \le x \le \frac{i}{m}, \\[2ex] \dfrac{D(x,\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})}, & \frac{i}{m} \le x \le \frac{i+1}{m}, \end{cases} & i = 1,\dots,m-1, \\[2ex] \Delta_m(x) = \dfrac{D(\frac{m-1}{m},x)}{D(\frac{m-1}{m},1)}, & \frac{m-1}{m} \le x \le 1. \end{cases}$$

Now (3.25), (3.26), and (3.27) define an almost tridiagonal system for the unkmowns $u(\frac{i}{m})$, $i = 0,\dots,m$. If we eliminate $u(\frac{2}{m})$ in the first equation by the second equation and $u(\frac{m-2}{m})$ in the last equation by the last but one equation, we get a tridiagonal system. To compute intermediate values of u consider the determinant formulation of the sum equation, described above, at the points

$\frac{i}{m} < \frac{j}{n} < \frac{i+1}{m}$, we get

$$(3.29)\quad y(\tfrac{j}{n}) + u(\tfrac{j}{n}) = \frac{D(\frac{j}{n},\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})}\left[y(\tfrac{i}{m}) + u(\tfrac{i}{m})\right] +$$

$$+ \frac{D(\frac{i}{m},\frac{j}{n})}{D(\frac{i}{m},\frac{i+1}{m})}\left[y(\tfrac{i+1}{m}) + u(\tfrac{i+1}{m})\right] +$$

$$+ \frac{D(\frac{j}{n},\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} \sum_{k=ri+1}^{j} \frac{1}{n} D(\tfrac{i}{m},\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) +$$

$$+ \frac{D(\frac{i}{m},\frac{j}{n})}{D(\frac{i}{m},\frac{i+1}{m})} \sum_{k=j+1}^{r(i+1)} \frac{1}{n} D(\tfrac{k}{n},\tfrac{i+1}{m}) f(\tfrac{k}{n},y(\tfrac{k}{n})) +$$

$$+ \frac{1}{12n^2}\left[\frac{f(\frac{j}{n},y(\frac{j}{n}))}{p(\frac{j}{n})} - \frac{D(\frac{j}{n},\frac{i+1}{m}) f(\frac{i}{m},y(\frac{i}{m}))}{D(\frac{i}{m},\frac{i+1}{m}) p(\frac{i}{m})} - \frac{f(\frac{i+1}{m},y(\frac{i+1}{m}))}{D(\frac{i}{m},\frac{i+1}{m}) p(\frac{i+1}{m})} D(\tfrac{i}{m},\tfrac{j}{n})\right],$$

for $j = ri+1,\ldots,r(i+1)-1$, $i = 0,\ldots,m-1$. We get a new y by

$$y(\tfrac{k}{n}) := y(\tfrac{k}{n}) + u(\tfrac{k}{n}), \quad k = 0,\ldots,n.$$

Thus the linear system defined by (3.25), (3.26), and (3.27) followed by equation (3.29) define a Newton-like method for the $O(h^4)$ approximation (3.20), (3.22), and (3.23) of the boundary value problem (2.1).

If we use the trapezoidal rule to obtain (3.25), (3.26), and (3.27), instead of the trapezoidal rule with "end correction", we get

$$(3.30)\quad \left[\frac{a(\frac{1}{m})}{a(0)D(0,\frac{1}{m})} - \frac{1}{2m} f_y'(0,y(0))\right] u(0) + \frac{-u(\frac{1}{m})}{D(0,\frac{1}{m})} =$$

$$= -\left\{\frac{a(\frac{1}{m})}{a(0)D(0,\frac{1}{m})} y(0) + \frac{-y(\frac{1}{m})}{D(0,\frac{1}{m})} - \frac{1}{3n}\left[f(0,y(0)) +\right.\right.$$

$$\left.\left. + \frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})} f(\tfrac{1}{n},y(\tfrac{1}{n}))\right] - \frac{1}{n}\sum_{k=1}^{r-1} \Delta_0(\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) -\right.$$

$$- \frac{1}{12n^2 D(0,\frac{1}{m})D(0,\frac{2}{n})}\left[- D(\tfrac{2}{n},\tfrac{1}{m})\frac{f(0,y(0))}{p(0)} + D(0,\tfrac{1}{m})\frac{f(\frac{2}{n},y(\frac{2}{n}))}{p(\frac{2}{n})} - \right.$$

$$\left. - D(0,\tfrac{2}{n})\frac{f(\frac{1}{m},y(\frac{1}{m}))}{p(\frac{1}{m})}\right] - \frac{c}{a(0)B_1(b)}\Big\}, \quad \text{if} \quad a_{12} > 0,$$

$$(3.31)\quad \frac{-u(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\frac{D(\frac{i-1}{m},\frac{i+1}{m})}{D(\frac{i-1}{m},\frac{i}{m})D(\frac{i}{m},\frac{i+1}{m})} - \frac{f'_y(\frac{i}{m},y(\frac{i}{m}))}{m}\right]u(\tfrac{i}{m}) + \frac{-u(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} =$$

$$= -\Big\{\frac{-y(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \frac{D(\frac{i-1}{m},\frac{i+1}{m})}{D(\frac{i-1}{m},\frac{i}{m})D(\frac{i}{m},\frac{i+1}{m})}y(\tfrac{i}{m}) + \frac{-y(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -$$

$$- \frac{1}{n}\sum_{k=r(i-1)+1}^{r(i+1)-1}\Delta_i(\tfrac{k}{n})f(\tfrac{k}{n},y(\tfrac{k}{n})) - \frac{1}{12n^2}\left[\frac{-f(\frac{i-1}{m},y(\frac{i-1}{m}))}{D(\frac{i-1}{m},\frac{i}{m})p(\frac{i-1}{m})} + \right.$$

$$\left. + \frac{D(\frac{i-1}{m},\frac{i+1}{m})}{D(\frac{i-1}{m},\frac{i}{m})D(\frac{i}{m},\frac{i+1}{m})}\frac{f(\frac{i}{m},y(\frac{i}{m}))}{p(\frac{i}{m})} - \frac{-f(\frac{i+1}{m},y(\frac{i+1}{m}))}{D(\frac{i}{m},\frac{i+1}{m})p(\frac{i+1}{m})}\right]\Big\}, \quad i = 1,\ldots,m-1,$$

and

$$(3.32)\quad \frac{-u(\frac{m-1}{m})}{D(\frac{m-1}{m},1)} + \left[\frac{b(\frac{m-1}{m})}{b(1)D(\frac{m-1}{m},1)} - \frac{f'_y(1,y(1))}{2m}\right]u(1) =$$

$$= -\Big\{\frac{-y(\frac{m-1}{m})}{D(\frac{m-1}{m},1)} + \frac{b(\frac{m-1}{m})}{b(1)D(\frac{m-1}{m},1)}y(1) - \frac{1}{3n}\left[f(1,y(1)) + \right.$$

$$\left. + \frac{D(\frac{n-2}{n},\frac{n-1}{n})}{D(\frac{n-2}{n},1)}f(\tfrac{n-1}{n},y(\tfrac{n-1}{n}))\right] - \frac{1}{n}\sum_{k=n-r+1}^{n-1}\Delta_m(\tfrac{k}{n})f(\tfrac{k}{n},y(\tfrac{k}{n})) -$$

$$- \frac{1}{12n^2 D(\frac{m-1}{m},1)D(\frac{n-2}{n},1)}\left[- D(\tfrac{n-2}{n},1)\frac{f(\frac{m-1}{m},y(\frac{m-1}{m}))}{p(\frac{m-1}{m})} + \right.$$

$$\left. + D(\tfrac{m-1}{m},1)\frac{f(\frac{n-2}{n},y(\frac{n-2}{n}))}{p(\frac{n-2}{n})} - D(\tfrac{m-1}{m},1)\frac{f(1,y(1))}{p(1)}\right] -$$

$$\left. - \frac{d}{b(1)B_2(a)}\right\}, \quad \text{if} \quad b_{22} > 0.$$

Now (3.30), (3.31), and (3.32) followed by equation (3.29) define

a Newton-like method for the $O(h^4)$ approximation (3.20), (3.22), and (3.23) of the boundary value problem (2.1). However for $m = n$ this method will still be a Newton-like method while (3.25), (3.26), and (3.27) give Newton's method for $m = n$. In chapter 9 example 9.2 numerical experiments have been done on these two Newton-like methods. We can see that the rate of convergence is almost the same for both methods, even for Newton's method. Since the matrix of (3.30), (3.31), and (3.32) is more simple to compute than the matrix of (3.25), (3.26), and (3.27) we put the Newton-like method defined by (3.30), (3.31), (3.32) followed by (3.29) in favor of the other method. The algorithm defined by (3.14) and (3.15), for evaluation of the sums, can also be used for these Newton-like methods.

Another Newton-like method is defined by Newton's method where f'_y only is evaluated on the grid G_m and f'_y is defined on $G_n \setminus G_m$ by linear interpolation. In example 9.3 of chapter 9 we see that the rate of convergence of this method is higher than for the Newton-lile methods given above. However the matrix of this method will be of order $n+1$ and there isn't any obvious way of doing the computations in parallel.

4. NEWTON-LIKE METHODS FOR BOUNDARY VALUE PROBLEMS WITH $q = 0$

In this chapter we will consider a special boundary value problem. In general Green's function for the boundary value problem (2.1) is not known. However if we write the differential equation in (2.1) as

$$L_p y = (py')' = f(t,y(t)) - q(t)y(t),$$

then Green's function for L_p, subject to the boundary conditions in (2.1), is easy to obtain. Thus we consider the boundary value problem

$$(4.1)\quad \begin{cases} L_p y = (py')' = f(t,y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

where p, p', and f are continuous and $p > 0$ on the interval $[0,1]$, and $a_{11}, a_{12}, b_{21}, b_{22} \geq 0$, $a_{11} + b_{21} > 0$, $a_{11} + a_{12} > 0$, $b_{21} + b_{22} > 0$. We also assume that $f'_y(t,y) \in C([0,1] \times R)$.

It is easy to see that all eigenvalues μ of the eigenvalue problem

$$(4.2)\quad \begin{cases} (py')' + \mu y = 0 \\ B_1 y = B_2 y = 0 \end{cases}$$

are greater than zero (see Aulin [10]). Thus Green's function for (4.1) exists and we have

$$(4.3)\quad G(x,t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

where

$$(4.4)\quad \begin{cases} a(x) = [a_{11}p(0)P(x) + a_{12}]/D_p \\ b(x) = b_{21}p(1)[P(x) - P(1)] - b_{22}, \end{cases}$$

and

$$(4.5)\quad \begin{cases} P(x) = \int_0^x \frac{dt}{p(t)}, & 0 \le x \le 1, \\ D_p = a_{11}b_{21}p(0)p(1)P(1) + a_{11}b_{22}p(0) + a_{12}b_{21}p(1) > 0. \end{cases}$$

The integral equation formulation of the boundary value problem (4.1) is then given by

$$(4.6)\quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \le x \le 1.$$

Let G_n denote the grid $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$. Then the determinant formulation of (4.6) on the grid G_n is

$$(4.7)\quad \begin{cases} \left[\frac{1}{D(0,x_1)} - \frac{a_{11}p(0)}{a_{12}}\right]y(0) + \frac{-y(x_1)}{D(0,x_1)} = \int_0^{x_1} \Delta_0(t)f(t,y(t))dt - \\ - \frac{cp(0)}{a_{12}}, \quad \text{if } a_{12} > 0, \\ \frac{-y(x_{i-1})}{D(x_{i-1},x_i)} + \left[\frac{1}{D(x_{i-1},x_i)} + \frac{1}{D(x_i,x_{i+1})}\right]y(x_i) + \frac{-y(x_{i+1})}{D(x_i,x_{i+1})} = \\ = \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t)f(t,y(t))dt, \quad i = 1,\ldots,n-1, \\ \frac{-y(x_{n-1})}{D(x_{n-1},1)} + \left[\frac{1}{D(x_{n-1},1)} - \frac{b_{21}p(1)}{b_{22}}\right]y(1) = \int_{x_{n-1}}^1 \Delta_n(t)f(t,y(t))dt - \\ - \frac{dp(1)}{b_{22}}, \quad \text{if } b_{22} > 0, \end{cases}$$

where we have used (2.7), (2.8), and

$$(4.8)\quad \begin{cases} \frac{a(x_1)}{a(0)D(0,x_1)} = \frac{1}{D(0,x_1)} - \frac{a_{11}p(0)}{a_{12}} \\ \frac{b(x_{n-1})}{b(1)D(x_{n-1},1)} = \frac{1}{D(x_{n-1},1)} - \frac{b_{21}p(1)}{b_{22}}. \end{cases}$$

We also have

$$
(4.9)\quad \begin{cases}
D(x_i,x_{i+1}) = -\displaystyle\int_{x_i}^{x_{i+1}} dt/p(t), & i = 0,\dots,n-1,\\[2ex]
D(x_{i-1},x_{i+1}) = D(x_{i-1},x_i) + D(x_i,x_{i+1}), & i = 1,\dots,n-1,\\[2ex]
\Delta_0(x) = D(x,x_1)/D(0,x_1), & 0 \le x \le x_1\\[2ex]
\Delta_i(x) = \begin{cases} D(x_{i-1},x)/D(x_{i-1},x_i), & x_{i-1} \le x \le x_i,\\ D(x,x_{i+1})/D(x_i,x_{i+1}), & x_i \le x \le x_{i+1}, \end{cases} & i = 1,\dots,n-1,\\[2ex]
\Delta_n(x) = D(x_{n-1},x)/D(x_{n-1},1), & x_{n-1} \le x \le 1.
\end{cases}
$$

If we approximate the integral in (4.6) by the trapezoidal rule on the grid G_n we get

$$
(4.10)\quad y(x_i) = \sum_{k=0}^{n} w_k G(x_i,x_k) f(x_k,y(x_k)) + \frac{cb(x_i)}{B_1(b)} + \frac{da(x_i)}{B_2(a)},
$$

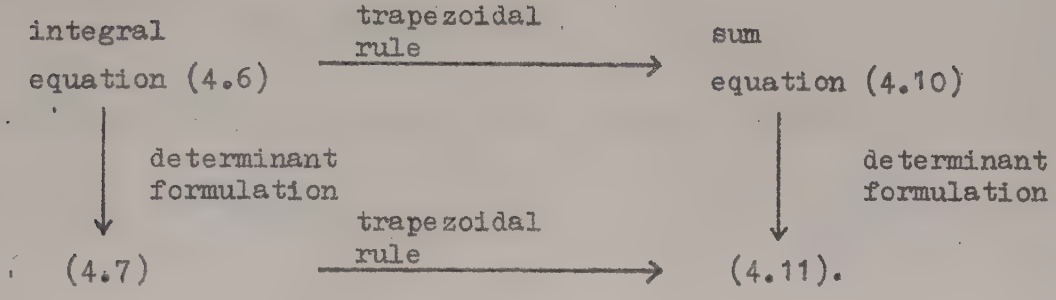
for $i = 0,\dots,n$, where

$$
w_k = \begin{cases}
\dfrac{x_1}{2}, & k = 0,\\[2ex]
\dfrac{x_{i+1} - x_{i-1}}{2}, & k = 1,\dots,n-1,\\[2ex]
\dfrac{1 - x_{n-1}}{2}, & k = n.
\end{cases}
$$

Since the trapezoidal rule commutes with the determinant formulation (see Aulin [10]) we obtain the determinant formulation of (4.10) by approximating the integrals in (4.7) by the trapezoidal rule, we get

$$
(4.11)\begin{cases}
\left[\dfrac{1}{D(0,x_1)} - \dfrac{a_{11}p(0)}{a_{12}}\right]y(0) + \dfrac{-y(x_1)}{D(0,x_1)} = w_0 f(0,y(0)) - \dfrac{cp(0)}{a_{12}}, \\
\text{if } a_{12} > 0, \\
\dfrac{-y(x_{i-1})}{D(x_{i-1},x_i)} + \left[\dfrac{1}{D(x_{i-1},x_i)} + \dfrac{1}{D(x_i,x_{i+1})}\right]y(x_i) + \dfrac{-y(x_{i+1})}{D(x_i,x_{i+1})} = \\
= w_i f(x_i,y(x_i)), \quad i = 1,\dots,n-1, \\
\dfrac{-y(x_{n-1})}{D(x_{n-1},1)} + \left[\dfrac{1}{D(x_{n-1},1)} - \dfrac{b_{21}p(1)}{b_{22}}\right]y(1) = w_n f(1,y(1)) - \dfrac{dp(1)}{b_{22}}, \\
\text{if } b_{22} > 0.
\end{cases}
$$

Hence we have the following commutative diagram



Consider the integrals

$$P(x_i) = \int_0^{x_i} dt/p(t), \quad i = 0,\dots,n.$$

We have

$$\begin{cases} P(0) = 0, \\ P(x_i) = -\left[D(0,x_1) + \dots + D(x_{i-1},x_i)\right], \quad i = 1,\dots,n. \end{cases}$$

If a fixed rule of integration is applied to each of the subintervals $[x_{i-1},x_i]$, $i = 1,\dots,n$, to approximate $D(x_{i-1},x_i)$ we get a compound rule for the approximation of $P(x_i)$, $i = 1,\dots,n$. Thus let

$$D(x_{i-1},x_i) \approx \hat{D}(x_{i-1},x_i), \quad i = 1,\dots,n,$$

where $\hat{D}$ denotes the approximation of D by such a rule of integration. Put

$$\begin{cases} \hat{P}(0) = 0 \\ \hat{P}(x_i) = -[\hat{D}(0,x_1) + \dots + \hat{D}(x_{i-1},x_i)], \quad i = 1,\dots,n, \\ \hat{a}(x_i) = [a_{11}p(0)\hat{P}(x_i) + a_{12}]/\hat{D}_p, \quad i = 0,\dots,n, \\ \hat{b}(x_i) = b_{11}p(1)[\hat{P}(x_i) - \hat{P}(1)] - b_{22}, \quad i = 0,\dots,n, \\ \hat{D}_p = a_{11}b_{21}p(0)p(1)\hat{P}(1) + a_{11}b_{22}p(0) + a_{12}b_{21}p(1), \end{cases}$$

then we have the following approximation of Green's function in (4.10)

$$\hat{G}(x_i,x_k) = \begin{cases} \hat{b}(x_i)\hat{a}(x_k), & i \geq k, \\ \hat{a}(x_i)\hat{b}(x_k), & i \leq k, \end{cases}$$

and the following approximation of the integral equation

$$(4.12) \quad y(x_i) = \sum_{k=0}^{n} w_k \hat{G}(x_i,x_k) f(x_k,y(x_k)) + \frac{c\hat{b}(x_i)}{B_1(\hat{b})} + \frac{d\hat{a}(x_i)}{B_2(\hat{a})},$$

for $i = 0,\dots,n$, where

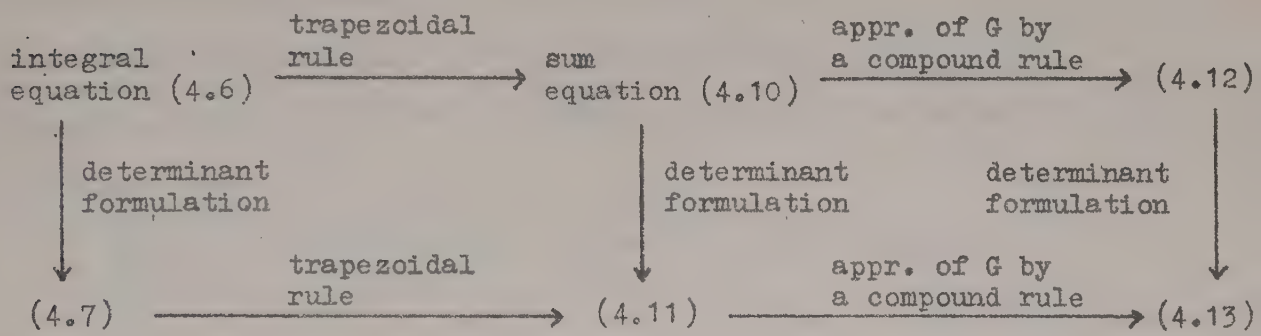
$$\frac{1}{B_1(\hat{b})} = \begin{cases} -\hat{a}(0)p(0)/a_{12}, & \text{if} \quad a_{12} > 0, \\ 1/(a_{11}\hat{b}(0)), & \text{if} \quad a_{12} = 0, \end{cases}$$

$$\frac{1}{B_2(\hat{a})} = \begin{cases} -\hat{b}(1)p(1)/b_{22}, & \text{if} \quad b_{22} > 0, \\ 1/(b_{21}\hat{a}(1)), & \text{if} \quad b_{22} = 0. \end{cases}$$

It is easy to see that the determiant formulation of (4.12) and the approximation of $D(x_{i-1},x_i)$ by $\hat{D}(x_{i-1},x_i)$ in (4.11) gives the same equation. Thus the determinant formulation of (4.12) is

$$
(4.13)\quad \begin{cases}
\left[\dfrac{1}{\hat{D}(0,x_1)} - \dfrac{a_{11}p(0)}{a_{12}}\right] y(0) + \dfrac{-y(x_1)}{\hat{D}(0,x_1)} = w_0 f(0,y(0)) - \dfrac{cp(0)}{a_{12}}, \\
\text{if } a_{12} > 0, \\
\dfrac{-y(x_{i-1})}{\hat{D}(x_{i-1},x_i)} + \left[\dfrac{1}{\hat{D}(x_{i-1},x_i)} + \dfrac{1}{\hat{D}(x_i,x_{i+1})}\right] y(x_i) + \dfrac{-y(x_{i+1})}{\hat{D}(x_i,x_{i+1})} = \\
= w_i f(x_i,y(x_i)), \quad i = 1,\ldots,n-1, \\
\dfrac{-y(x_{n-1})}{\hat{D}(x_{n-1},1)} + \left[\dfrac{1}{\hat{D}(x_{n-1},1)} - \dfrac{b_{21}p(1)}{b_{22}}\right] y(1) = w_n f(1,y(1)) - \dfrac{dp(1)}{b_{22}}, \\
\text{if } b_{22} > 0.
\end{cases}
$$

Now we can extend the first commutative diagram, given above, to the following commutative diagram



Remark.

For simplicity we have used the same symbol y for the unknowns in equations (4.6) - (4.13). We note that (4.6) and (4.7) have the same solution since the systems are equivalent. In the same way the two systems (4.10) and (4.11) have the same solution just as (4.12) and (4.13). Thus to be exact we should have used three symbols for the unknowns. We also mention that a diagram of mappings is commutative if any two sequences of maps that start and end at the same sets produce the same composite map.

By the commutative property Newton-like methods for the sum equation (4.12) have the same form as Newton-like methods for (4.10). We only have to replace $D(x_{i-1},x_i)$ by $\hat{D}(x_{i-1},x_i)$, $i = 1,\ldots,n$, in the Newton-like methods for (4.10).

Consider the grid

$$G_n: 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$$

and the subgrid (m n)

$$G_m: 0 = x_{n_0} < x_{n_1} < \ldots < x_{n_m} = 1,$$

where n_i are integers and $0 \le n_i \le n$, $i = 0,\ldots,m$. Put $t_i = x_{n_i}$, $i = 0,\ldots,m$. If we put $q = 0$ in (2.1) and approximate the integral equation formulation by the trapezoidal rule then the Newton-like method defined by (3.5), (3.7), (3.9), and (3.12) is also a Newton-like method for the boundary value problem (4.1). According to Pohl [15] we now write this Newton-like method in the following form

$$\left\{\begin{array}{l}
\left[\dfrac{1}{D(0,t_1)} - \dfrac{a_{11}p(0)}{a_{12}}\right]u(0) + \dfrac{-u(t_1)}{D(0,t_1)} - \omega_0 f'_y(0,y(0))u(0) = \\[2ex]
= -\left\{\dfrac{1}{D(0,t_1)}[y(0) - y(t_1)] + \dfrac{p(0)}{a_{12}}[c - a_{11}y(0)] - \right. \\[2ex]
\left. - \displaystyle\sum_{k=0}^{n_1-1} w_k \Delta_0(x_k) f(x_k,y(x_k))\right\}, \quad \text{if } a_{12} > 0, \\[2ex]
\dfrac{-u(t_{i-1})}{D(t_{i-1},t_i)} + \left[\dfrac{1}{D(t_{i-1},t_i)} + \dfrac{1}{D(t_i,t_{i+1})}\right]u(t_i) + \dfrac{-u(t_{i+1})}{D(t_i,t_{i+1})} - \\[2ex]
- \omega_i f'_y(t_i,y(t_i))u(t_i) = \\[2ex]
= -\left\{\left[\dfrac{1}{D(t_{i-1},t_i)}[y(t_i) - y(t_{i-1})] + \dfrac{1}{D(t_i,t_{i+1})}[y(t_i) - y(t_{i+1})]\right]\right.
\end{array}\right. \tag{4.14}$$

$$\left\{ - \sum_{k=n_{i-1}+1}^{n_{i+1}-1} w_k \Delta_i(x_k) f(x_k, y(x_k)) \right\}, \quad i = 1,\ldots,m-1,$$

$$\frac{- u(t_{m-1})}{D(t_{m-1},1)} + \left[\frac{1}{D(t_{m-1},1)} - \frac{b_{21} p(1)}{b_{22}} \right] u(1) - \omega_n f'_y(1, y(1)) u(1) =$$

$$= - \left\{ \frac{1}{D(t_{m-1},1)} [y(1) - y(t_{m-1})] + \frac{p(1)}{b_{22}} [d - b_{21} y(1)] - \right.$$

$$\left. - \sum_{k=n_{m-1}+1}^{n} w_k \Delta_m(x_k) f(x_k, y(x_k)) \right\}, \quad \text{if} \quad b_{22} > 0,$$

where

$$\omega_i = \begin{cases} t_1/2, & i = 0, \\ (t_{i+1} - t_{i-1})/2, & i = 1,\ldots,m-1, \\ (1 - t_{m-1})/2, & i = m; \end{cases}$$

$$w_k = \begin{cases} x_1/2, & k = 0, \\ (x_{k+1} - x_{k-1})/2, & k = 1,\ldots,n-1, \\ (1 - x_{n-1})/2, & k = n, \end{cases}$$

$$\begin{cases} \Delta_0(x) = D(x, t_1)/D(0, t_1), & 0 \le x \le t_1, \\ \Delta_i(x) = \begin{cases} D(t_{i-1}, x)/D(t_{i-1}, t_i), & t_{i-1} \le x \le t_i, \\ D(x, t_{i+1})/D(t_i, t_{i+1}), & t_i \le x \le t_{i+1}, \end{cases} & i = 1,\ldots,m-1, \\ \Delta_m(x) = D(t_{m-1}, x)/D(t_{m-1}, 1), & t_{m-1} \le x \le 1. \end{cases}$$

Intermediate values are given by the following equations

$$(4.15) \quad y(x_j) + u(x_j) = \frac{D(x_j, t_{i+1})}{D(t_i, t_{i+1})} [y(t_i) + u(t_i)] +$$

$$+ \frac{D(t_i, x_j)}{D(t_i, t_{i+1})} [y(t_{i+1}) + u(t_{i+1})] +$$

$$+\frac{D(x_j,t_{i+1})}{D(t_i,t_{i+1})}\sum_{k=n_i+1}^{j} w_k D(t_i,x_k) f(x_k,y(x_k)) +$$

$$+\frac{D(t_i,x_j)}{D(t_i,t_{i+1})}\sum_{k=j+1}^{n_{i+1}} w_k D(x_k,t_{i+1}) f(x_k,y(x_k)),$$

for $j = n_i+1,\dots, n_{i+1}-1$, $i = 0,\dots,m-1$. We get a new y by

$$y(x_j) := y(x_j) + u(x_j), \quad j = 0,\dots,n.$$

This Newton-like method is an $O(h^2)$ $(h = \max_i x_{i+1} - x_i)$ method for the boundary value problem (4.1) (see Aulin [10]).

The sums in (4.14) and (4.15) can be evaluated by the following algorithm

$$(4.16)\begin{cases} SU(i,0) = 0; \\ SU(i,k+1) = SU(i,k) + w_{n_i+k+1} D(t_i,x_{n_i+k+1}) f(x_{n_i+k+1},y(x_{n_i+k+1})); \\ k = 0,1,\dots,n_{i+1} - n_i - 1, \end{cases}$$

$$(4.17)\begin{cases} SD(i,n_{i+1}-n_i) = 0; \\ SD(i,n_{i+1}-n_i-k-1) = SD(i,n_{i+1}-n_i-k) + \\ + w_{n_{i+1}-k-1}\left[D(t_i,t_{i+1}) - D(t_i,x_{n_{i+1}-k-1})\right] f(x_{n_{i+1}-k-1},y(x_{n_{i+1}-k-1})); \\ k = 0,1,\dots,n_{i+1}-n_i-1, \end{cases}$$

for $i = 0,\dots,m-1$. Then we have

$$(4.18)\quad \sum_{k=n_{i-1}+1}^{n_{i+1}-1} w_k \Delta_i(x_k) f(x_k,y(x_k)) =$$

$$= SU(i-1,n_i-n_{i-1})/D(t_{i-1},t_i) + SD(i,1)/D(t_i,t_{i+1}),$$

for $i = 1,\dots,m-1$,

$$(4.19)\quad \sum_{k=0}^{n_1-1} w_k \Delta_0(x_k) f(x_k, y(x_k)) = SD(0,0)/D(0,t_1),$$

and

$$(4.20)\quad \sum_{k=n_{m-1}+1}^{n} w_k \Delta_m(x_k) f(x_k, y(x_k)) = SU(m-1, n-n_{m-1})/D(t_{m-1}, 1).$$

By SU and SD equation (4.15) can be written as

$$(4.21)\quad y(x_j) + u(x_j) = \left[1 - \frac{D(t_i,x_j)}{D(t_i,t_{i+1})}\right]\left[y(t_i) + u(t_i)\right] +$$

$$+ \frac{D(t_i,x_j)}{D(t_i,t_{i+1})}\left[y(t_{i+1}) + u(t_{i+1})\right] +$$

$$+ \left[1 - \frac{D(t_i,x_j)}{D(t_i,t_{i+1})}\right] SU(i, j-n_i) + \frac{D(t_i,x_j)}{D(t_i,t_{i+1})}\, SD(i, j+1-n_j),$$

for $j = n_i+1,\ldots,n_{i+1}-1$, $i = 0,\ldots,m-1$. Here we have used the property

$$D(x_j,t_{i+1}) = D(t_i,t_{i+1}) - D(t_i,x_j), \quad t_i \le x_j \le t_{i+1}.$$

We will now consider this Newton-like method when $D(x_i,x_{i+1})$, $i = 0,\ldots,n-1$, are replaced by $\hat{D}(x_i,x_{i+1})$, $i = 0,\ldots,n-1$, where $\hat{D}$ denotes the approximation of D by the midpoint rule. In this case we will get an $O(h^2)$-method if and only if the grid G_n is equidistant (see Aulin [10]). Let m,n, and r be positive integers and $n = rm$. Then m and n define two grids, G_m and G_n, on $[0,1]$ with equidistant points. We get $x_k = k/n$, $k = 0,\ldots,n$, and $t_i = i/m$, $i = 0,\ldots,m$. Consider (4.14) and (4.15) and put

$$\begin{cases} D(t_i,t_i) = 0, \\ D(t_i,x_j) = D(t_i,x_{j-1}) + 1/(np(\frac{j-\frac{1}{2}}{n})), \quad j = ri+1,\ldots, r(i+1), \\ D(x_j,t_{i+1}) = D(t_i,t_{i+1}) - D(t_i,x_j), \quad j = ri,\ldots,r(i+1), \end{cases}$$

for $i = 0,\ldots,m-1$. Then (4.14) followed by (4.15) define a Newton-like method for the numerical solution of boundary value problem (4.1).

To get an $O(h^4)$-method for the boundary value problem (4.1) let $q = 0$ in the boundary value problem (2.1). Then (3.30), (3.31), and (3.32) followed by equation (3.29) define a Newton-like method for the numerical solution of the boundary value problem (4.1) (we will not consider the system defined by (3.25), (3.26), and (3.27)). We now write these equations in the following form

$$(4.22)\quad \left[\frac{1}{D(0,\frac{1}{m})} - \frac{a_{11}p(0)}{a_{12}}\right]u(0) + \frac{-u(\frac{1}{m})}{D(0,\frac{1}{m})} - \frac{1}{2m} f'_y(0,y(0))u(0) =$$

$$= -\left\{\frac{1}{D(0,\frac{1}{m})}\left[y(0) - y(\tfrac{1}{m})\right] + \frac{p(0)}{a_{12}}\left[c - a_{11}y(0)\right] -\right.$$

$$- \frac{1}{3n}\left[f(0,y(0)) + \frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})} f(\tfrac{1}{n},y(\tfrac{1}{n}))\right] - \frac{1}{n}\sum_{k=1}^{r-1} \Delta_0(\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) -$$

$$- \frac{1}{12n^2 D(0,\frac{1}{m})D(0,\frac{2}{n})}\left[D(\tfrac{2}{n},\tfrac{1}{m})\left(\frac{f(\frac{2}{n},y(\frac{2}{n}))}{p(\frac{2}{n})} - \frac{f(0,y(0))}{p(0)}\right) +\right.$$

$$\left.\left. + D(0,\tfrac{2}{n})\left(\frac{f(\frac{2}{n},y(\frac{2}{n}))}{p(\frac{2}{n})} - \frac{f(\frac{1}{m},y(\frac{1}{m}))}{p(\frac{1}{m})}\right)\right]\right\}, \quad \text{if} \quad a_{12} > 0,$$

$$(4.23)\quad \frac{-u(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\frac{1}{D(\frac{i-1}{m},\frac{i}{m})} + \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right]u(\frac{i}{m}) + \frac{-u(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -$$

$$- \frac{1}{m} f'_y(\tfrac{i}{m},y(\tfrac{i}{m}))u(\tfrac{i}{m}) =$$

$$= -\left\{\frac{1}{D(\frac{i-1}{m},\frac{i}{m})}\left[y(\tfrac{i}{m}) - y(\tfrac{i-1}{m})\right] + \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\left[y(\tfrac{i}{m}) - y(\tfrac{i+1}{m})\right] -\right.$$

$$- \frac{1}{n}\sum_{k=r(i-1)+1}^{r(i+1)-1} \Delta_i(\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) -$$

$$- \frac{1}{12n^2}\left[\frac{1}{D(\frac{i-1}{m},\frac{i}{m})}\left(\frac{f(\frac{i}{m},y(\frac{i}{m}))}{p(\frac{i}{m})} - \frac{f(\frac{i-1}{m},y(\frac{i-1}{m}))}{p(\frac{i-1}{m})}\right) +\right.$$

$$\left.\left. + \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\left(\frac{f(\frac{i}{m},y(\frac{i}{m}))}{p(\frac{i}{m})} - \frac{f(\frac{i+1}{m},y(\frac{i+1}{m}))}{p(\frac{i+1}{m})}\right)\right]\right\}, \quad i = 1,\ldots,m-1,$$

$$(4.24)\quad \frac{-u(\frac{m-1}{m})}{D(\frac{m-1}{m},1)} + \left[\frac{1}{D(\frac{m-1}{m},1)} - \frac{b_{21}p(1)}{b_{22}}\right]u(1) - \frac{1}{2m} f'_y(1,y(1))u(1) =$$

$$= -\left\{\frac{1}{D(\frac{m-1}{m},1)}\left[y(1) - y(\tfrac{m-1}{m})\right] + \frac{p(1)}{b_{22}}\left[d - b_{21}y(1)\right] -\right.$$

$$- \frac{1}{3n}\left[f(1,y(1)) + \frac{D(\frac{n-2}{n},\frac{n-1}{n})}{D(\frac{n-2}{n},1)} f(\tfrac{n-1}{n},y(\tfrac{n-1}{n}))\right] -$$

$$- \frac{1}{n}\sum_{k=n-r+1}^{n-1} \Delta_m(\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) -$$

$$- \frac{1}{12n^2 D(\frac{m-1}{m},1)D(\frac{n-2}{n},1)}\left[D(\tfrac{n-2}{n},1)\left(\frac{f(\frac{n-2}{n},y(\frac{n-2}{n}))}{p(\frac{n-2}{n})} - \frac{f(\frac{m-1}{m},y(\frac{m-1}{m}))}{p(\frac{m-1}{m})}\right) +\right.$$

$$\left.\left. + D(\tfrac{m-1}{m},\tfrac{n-2}{n})\left(\frac{f(\frac{n-2}{n},y(\frac{n-2}{2}))}{p(\frac{n-2}{n})} - \frac{f(1,y(1))}{p(1)}\right)\right]\right\}, \quad \text{if} \quad b_{22} > 0,$$

and

$$(4.25)\quad y(\tfrac{j}{n}) + u(\tfrac{j}{n}) = \frac{D(\frac{j}{n},\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})}\left[y(\tfrac{i}{m}) + u(\tfrac{i}{m})\right] +$$

$$+ \frac{D(\frac{i}{m},\frac{j}{n})}{D(\frac{i}{m},\frac{i+1}{m})}\left[y(\tfrac{i+1}{m}) + u(\tfrac{i+1}{m})\right] +$$

$$+ \frac{D(\frac{j}{n},\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} \sum_{k=ri+1}^{j} \frac{1}{n} D(\tfrac{i}{m},\tfrac{k}{n}) f(\tfrac{k}{n},y(\tfrac{k}{n})) +$$

$$+ \frac{D(\frac{i}{m},\frac{j}{n})}{D(\frac{i}{m},\frac{i+1}{m})} \sum_{k=j+1}^{r(i+1)} \frac{1}{n} D(\tfrac{k}{n},\tfrac{i+1}{m}) f(\tfrac{k}{n},y(\tfrac{k}{n})) +$$

$$+ \frac{1}{12n^2}\left[\frac{D(\frac{j}{n},\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})}\left(\frac{f(\frac{j}{n},y(\frac{j}{n}))}{p(\frac{j}{n})} - \frac{f(\frac{i}{m},y(\frac{i}{m}))}{p(\frac{i}{m})}\right) +\right.$$

$$\left. + \frac{D(\frac{i}{m},\frac{j}{n})}{D(\frac{i}{m},\frac{i+1}{m})}\left(\frac{f(\frac{j}{n},y(\frac{j}{n}))}{p(\frac{j}{n})} - \frac{f(\frac{i+1}{m},y(\frac{i+1}{m}))}{p(\frac{i+1}{m})}\right)\right]\Big\},$$

for $j = ri+1,\ldots,r(i+1)-1$, $i = 0,\ldots,m-1$. We get a new y by

$$y(\tfrac{j}{n}) := y(\tfrac{j}{n}) + u(\tfrac{j}{n}), \quad j = 0,\ldots,n.$$

Thus the linear system defined by (4.22), (4.23), and (4.24) followed by equation (4.25) define a Newton-like method for the following $O(h^4)$ approximation of the boundary value problem (4.1) ($h = 1/n$)

$$(4.26)\quad \left[\frac{1}{D(0,\frac{1}{n})} - \frac{a_{11}p(0)}{a_{12}}\right]y(0) + \frac{-y(\frac{1}{n})}{D(0,\frac{1}{n})} = \frac{h}{3}\left[f(0,y(0)) +\right.$$

$$\left. + \frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})} f(\tfrac{1}{n},y(\tfrac{1}{n}))\right] - \frac{h^2}{12}\frac{D(\frac{1}{n},\frac{2}{n})}{D(0,\frac{2}{n})}\left[\frac{1}{D(0,\frac{1}{n})}\left(\frac{f(\frac{1}{n},y(\frac{1}{n}))}{p(\frac{1}{n})} - \frac{f(0,y(0))}{p(0)}\right) +\right.$$

$$\left. + \frac{1}{D(\frac{1}{n},\frac{2}{n})}\left(\frac{f(\frac{1}{n},y(\frac{1}{n}))}{p(\frac{1}{n})} - \frac{f(\frac{2}{n},y(\frac{2}{n}))}{p(\frac{2}{n})}\right)\right] - \frac{cp(0)}{a_{12}}, \quad \text{if } a_{12} > 0,$$

$$(4.27)\quad \frac{-y(\frac{i-1}{n})}{D(\frac{i-1}{n},\frac{i}{n})} + \left[\frac{1}{D(\frac{i-1}{n},\frac{i}{n})} + \frac{1}{D(\frac{i}{n},\frac{i+1}{n})}\right] y(\frac{i}{n}) + \frac{-y(\frac{i+1}{n})}{D(\frac{i}{n},\frac{i+1}{n})} =$$

$$= hf(\tfrac{i}{n},y(\tfrac{i}{n})) + \frac{h^2}{12}\left[\frac{1}{D(\frac{i-1}{n},\frac{i}{n})}\left(\frac{f(\frac{i}{n},y(\frac{i}{n}))}{p(\frac{i}{n})} - \frac{f(\frac{i-1}{n},y(\frac{i-1}{n}))}{p(\frac{i-1}{n})}\right) +\right.$$

$$\left. + \frac{1}{D(\frac{i}{n},\frac{i+1}{n})}\left(\frac{f(\frac{i}{n},y(\frac{i}{n}))}{p(\frac{i}{n})} - \frac{f(\frac{i+1}{n},y(\frac{i+1}{n}))}{p(\frac{i+1}{n})}\right)\right], \quad i = 1,\ldots,n-1,$$

$$(4.28)\quad \frac{-y(\frac{n-1}{n})}{D(\frac{n-1}{n},1)} + \left[\frac{1}{D(\frac{n-1}{n},1)} - \frac{b_{21}p(1)}{b_{22}}\right] y(1) =$$

$$= \frac{h}{3}\left[\frac{D(\frac{n-2}{n},\frac{n-1}{n})}{D(\frac{n-1}{n},1)}\, f(\tfrac{n-1}{n},y(\tfrac{n-1}{n})) + f(,y(1))\right] -$$

$$- \frac{h^2}{12}\frac{D(\frac{n-2}{n},\frac{n-1}{n})}{D(\frac{n-2}{n},1)}\left[\frac{1}{D(\frac{n-2}{n},\frac{n-1}{n})}\left(\frac{f(\frac{n-1}{n},y(\frac{n-1}{n}))}{p(\frac{n-1}{n})} - \frac{f(\frac{n-2}{n},y(\frac{n-2}{n}))}{p(\frac{n-2}{n})}\right) +\right.$$

$$\left. + \frac{1}{D(\frac{n-1}{n},1)}\left(\frac{f(\frac{n-1}{n},y(\frac{n-1}{n}))}{p(\frac{n-1}{n})} - \frac{f(1,y(1))}{p(1)}\right)\right] - \frac{dp(1)}{b_{22}}, \quad \text{if} \quad b_{22} > 0.$$

If $D(\frac{i}{n},\frac{i+1}{n})$, $i = 0,\ldots,n-1$, cannot be evaluated exactly then the approximation of D in (4.22), (4.23), (4.24), and (4.25) by Simpson's rule

$$(4.29)\quad D(\tfrac{i}{n},\tfrac{i+1}{n}) = -\frac{1}{6n}\left[\frac{1}{p(\frac{i}{n})} + \frac{4}{p(\frac{i+\frac{1}{2}}{n})} + \frac{1}{p(\frac{i+1}{n})}\right], \quad i = 0,\ldots,n-1,$$

will also give an $O(h^4)$-method for the boundary value problem (4.1) (see Aulin' [10]).

5. NEWTON-LIKE METHODS FOR BOUNDARY VALUE PROBLEMS WITH p AND q CONSTANT

We may assume that $p = 1$. Hence we will consider boundary value problems of the form

$$(5.1)\quad \begin{cases} y'' - q^2 y = f(t,y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

where f is continuous, $f'_y(t,y) \in C([0,1] \times R)$, and $a_{11}, a_{12}, b_{21}, b_{22} \geq 0$, $a_{11} + a_{12} > 0$, $b_{21} + b_{22} > 0$, $a_{11} + b_{21} > 0$.

With $q := iq$, the differential equation takes the form

$$(5.2)\quad y'' + q^2 y = f(t,y(t)), \quad 0 < t < 1,$$

but we will only consider the case (5.1) with $q \geq 0$.

We first assume that $q = 0$. Then we have the boundary value problem (4.1) with $p = 1$, and the Newton-like methods given in chapter 4 apply. However we are able to make some simplifications for these methods since $p(t) \equiv 1$. We will rewrite the Newton-like method defined by (4.14) and (4.15) for this problem on an equidistant grid. Thus let m, n, and r be positive integers and $n = rm$. Then m and n define two grids, G_m and G_n, on $[0,1]$ with equidistant points. We have

$$(5.3)\quad D\left(\frac{i}{n},\frac{i+1}{n}\right) = -\int_{\frac{i}{n}}^{\frac{i+1}{n}} dt = -\frac{1}{n}, \quad i = 0,\ldots,n-1,$$

or $D(x,t) = x - t$, $0 \leq x \leq t \leq 1$. Then the approximation of the determinant formulation of (5.1), with $q = 0$, by the trapezoidal rule on the grid G_n is

$$(5.4)\quad \begin{cases} -\left[1+\dfrac{a_{11}}{na_{12}}\right]y(0)+y\left(\dfrac{1}{n}\right)=\dfrac{1}{2n^2}f(0,y(0))-\dfrac{c}{na_{12}}, & \text{if } a_{12}>0,\\ y\left(\dfrac{i-1}{n}\right)-2y\left(\dfrac{i}{n}\right)+y\left(\dfrac{i+1}{n}\right)=\dfrac{1}{n^2}f\left(\dfrac{i}{n},y\left(\dfrac{i}{n}\right)\right), & i=1,\dots,n-1,\\ y\left(\dfrac{n-1}{n}\right)-\left[1+\dfrac{b_{21}}{nb_{22}}\right]y(1)=\dfrac{1}{2n^2}f(1,y(1))-\dfrac{d}{nb_{22}}, & \text{if } b_{22}>0. \end{cases}$$

By (4.14) and (4.15) a Newton-like method for (5.4) is given by the following iterative method.

$$(5.5)\quad \begin{cases} -\left[1+\dfrac{a_{11}}{ma_{12}}+\dfrac{1}{2m^2}f'_y(0,y(0))\right]u(0)+u\left(\dfrac{1}{m}\right)=\\ =-\left\{\left[y\left(\dfrac{1}{m}\right)-y(0)\right]+\dfrac{1}{ma_{12}}\left[c-a_{11}y(0)\right]-\dfrac{1}{2mn}f(0,y(0))-\right.\\ \left.-\dfrac{1}{mn}\sum_{k=1}^{r-1}\left(1-\dfrac{k}{r}\right)f\left(\dfrac{k}{n},y\left(\dfrac{k}{n}\right)\right)\right\}, \quad \text{if } a_{12}>0,\\ u\left(\dfrac{i-1}{m}\right)-2u\left(\dfrac{i}{m}\right)+u\left(\dfrac{i+1}{m}\right)-\dfrac{1}{m^2}f'_y\left(\dfrac{i}{m},y\left(\dfrac{i}{m}\right)\right)u\left(\dfrac{i}{m}\right)=\\ =-\left\{\left[y\left(\dfrac{i-1}{m}\right)-y\left(\dfrac{i}{m}\right)\right]+\left[y\left(\dfrac{i+1}{m}\right)-y\left(\dfrac{i}{m}\right)\right]-\right.\\ -\dfrac{1}{mn}\left[\sum_{k=1}^{r}\dfrac{k}{r}f\left(\dfrac{i-1}{m}+\dfrac{k}{n},y\left(\dfrac{i-1}{m}+\dfrac{k}{n}\right)\right)+\right.\\ \left.\left.+\sum_{k=1}^{r-1}\left(1-\dfrac{k}{r}\right)f\left(\dfrac{i}{m}+\dfrac{k}{n},y\left(\dfrac{i}{m}+\dfrac{k}{n}\right)\right)\right]\right\}, \quad i=1,\dots,m-1,\\ u\left(\dfrac{m-1}{m}\right)-\left[1+\dfrac{b_{21}}{mb_{22}}+\dfrac{1}{2m^2}f'_y(1,y(1))\right]u(1)=\\ =-\left\{\left[y\left(\dfrac{m-1}{m}\right)-y(1)\right]+\dfrac{1}{mb_{22}}\left[d-b_{21}y(1)\right]-\right.\\ \left.-\dfrac{1}{mn}\sum_{k=1}^{r-1}\left(1-\dfrac{k}{r}\right)f\left(\dfrac{m-1}{m}+\dfrac{k}{n},y\left(\dfrac{m-1}{m}+\dfrac{k}{n}\right)\right)-\dfrac{1}{2mn}f(1,y(1))\right\}, \text{ if } b_{22}>0, \end{cases}$$

and

$$(5.6)\quad y(\tfrac{i}{m}+\tfrac{j}{n}) + u(\tfrac{i}{m}+\tfrac{j}{n}) = (1-\tfrac{j}{r})\left[y(\tfrac{i}{m}) + u(\tfrac{i}{m})\right] +$$

$$+ \tfrac{j}{r}\left[y(\tfrac{i+1}{m}) + u(\tfrac{i+1}{m})\right] -$$

$$- \tfrac{1}{mn}(1-\tfrac{j}{r}) \sum_{k=1}^{j} \tfrac{k}{r} f(\tfrac{i}{m}+\tfrac{k}{n}, y(\tfrac{i}{m}+\tfrac{k}{n})) -$$

$$- \tfrac{1}{mn}\tfrac{j}{r} \sum_{k=j+1}^{r} (1-\tfrac{k}{r}) f(\tfrac{i}{m}+\tfrac{k}{n}, y(\tfrac{i}{m}+\tfrac{k}{n})), \quad j = 1,\ldots,r-1,$$

for $i = 0,\ldots,m-1$. We get a new y by

$$y(\tfrac{j}{n}) := y(\tfrac{j}{n}) + u(\tfrac{j}{n}), \quad j = 0,\ldots,n.$$

The sums in (5.5) and (5.6) can be evaluated by the following algorithm.

$$(5.7)\quad \begin{cases} SU(i,0) = 0; \\ SU(i,k+1) = SU(i,k) + (k+1) f(\tfrac{i}{m} + \tfrac{k+1}{n}, y(\tfrac{i}{m} + \tfrac{k+1}{n})); \\ k = 0,\ldots,r-1, \end{cases}$$

$$(5.8)\quad \begin{cases} SD(i,r) = 0; \\ SD(i,r-k-1) = SD(i,r-k) + (k+1) f(\tfrac{i+1}{m} - \tfrac{k+1}{n}, y(\tfrac{i+1}{m} - \tfrac{k+1}{n})); \\ k = 0,\ldots,r-2, \end{cases}$$

for $i = 0,\ldots,m-1$. Then we have

$$(5.9)\quad \tfrac{1}{mn} \sum_{k=1}^{r-1} (1-\tfrac{k}{r}) f(\tfrac{k}{n}, y(\tfrac{k}{n})) = SD(0,1)/n^2,$$

$$(5.10)\quad \tfrac{1}{mn}\left[\sum_{k=1}^{r} \tfrac{k}{r} f(\tfrac{i-1}{m}+\tfrac{k}{n}, y(\tfrac{i-1}{m}+\tfrac{k}{n})) + \sum_{k=1}^{r-1} (1-\tfrac{k}{r}) f(\tfrac{i}{m}+\tfrac{k}{n}, y(\tfrac{i}{m}+\tfrac{k}{n}))\right] =$$

$$= \left[SU(i-1,r) + SD(i,1)\right]/n^2, \quad i = 1,\ldots,m-1,$$

$$(5.11)\quad \tfrac{1}{mn} \sum_{k=1}^{r-1} (1-\tfrac{k}{r}) f(\tfrac{m-1}{m}+\tfrac{k}{n}, y(\tfrac{m-1}{m}+\tfrac{k}{n})) = SU(m-1,r-1)/n^2,$$

and

$$(5.12)\quad \frac{1}{mn}\Big[(1-\frac{j}{r})\sum_{k=1}^{j}\frac{k}{r}\,f(\frac{i}{m}+\frac{k}{n},y(\frac{i}{m}+\frac{k}{n}))+$$

$$+\frac{j}{r}\sum_{k=j+1}^{r}(1-\frac{k}{r})f(\frac{i}{m}+\frac{k}{n},y(\frac{i}{m}+\frac{k}{n}))\Big]=$$

$$=[(r-j)SU(i,j)+jSD(i,j+1)]/(rm^2),\quad j=1,\ldots,r-1,$$

for $i=0,\ldots,m-1$.

Newton-like methods for the numerical solution of boundary value problems of the form (5.1) with $q=0$ have also been given in Aulin [4], [5].

We will now assume that $q>0$ in (5.1). Green's function for this problem is

$$G(x,t)=\begin{cases}b(x)a(t), & x\geq t,\\ a(x)b(t), & x\leq t,\end{cases}$$

where

$$\begin{cases}a(x)=\Big[\dfrac{a_{11}}{q}\sinh(qx)+a_{12}\cosh(qx)\Big]/D_q,\\ b(x)=\dfrac{b_{21}}{q}\sinh(q(x-1))-b_{22}\cosh(q(x-1)),\\ D_q=\dfrac{a_{11}b_{21}}{q}\sinh(q)+a_{11}b_{22}\cosh(q)+a_{12}b_{21}\cosh(q)+\\ \quad+qa_{12}b_{22}\sinh(q)>0,\end{cases}$$

(see Aulin [10]). We get

$$(5.13)\quad D(\frac{i}{n},\frac{i+1}{n})=-\frac{\sinh(\frac{q}{n})}{q}<0,\quad i=0,\ldots,n-1,$$

or $D(x,t)=\sinh(q(x-t))/q$, $0\leq x\leq t\leq 1$. Then the approximation of the determinant formulation of (5.1), with $q>0$, by the trapezoidal rule on the grid G_n is

$$
(5.14)\begin{cases}
-\left[\cosh\left(\frac{q}{n}\right)+\frac{a_{11}\sinh\left(\frac{q}{n}\right)}{qa_{12}}\right]y(0)+y\left(\frac{1}{n}\right)=\frac{\sinh\left(\frac{q}{n}\right)}{2qn}f(0,y(0))- \\
-\frac{c\sinh\left(\frac{q}{n}\right)}{qa_{12}}, \quad \text{if} \quad a_{12}>0, \\
y\left(\frac{i-1}{n}\right)-2\cosh\left(\frac{q}{n}\right)y\left(\frac{i}{n}\right)+y\left(\frac{i+1}{n}\right)=\frac{\sinh\left(\frac{q}{n}\right)}{qn}f\left(\frac{i}{n},y\left(\frac{i}{n}\right)\right),\ i=1,\ldots,n-1, \\
y\left(\frac{n-1}{n}\right)-\left[\cosh\left(\frac{q}{n}\right)+\frac{b_{21}\sinh\left(\frac{q}{n}\right)}{qb_{22}}\right]y(1)=\frac{\sinh\left(\frac{q}{n}\right)}{2qn}f(1,y(1))- \\
-\frac{d\sinh\left(\frac{q}{n}\right)}{qb_{22}}, \quad \text{if} \quad b_{22}>0.
\end{cases}
$$

By (3.5), (3.7), (3.9), and (3.12) a Newton-like method for (5.14) is given by the following iterative method.

$$
(5.15)\begin{cases}
-\left[\cosh\left(\frac{q}{m}\right)+\frac{a_{11}\sinh\left(\frac{q}{m}\right)}{qa_{12}}\right]u(0)+u\left(\frac{1}{m}\right)-\frac{\sinh\left(\frac{q}{m}\right)}{2qm}f_y'(0,y(0))u(0)= \\
=-\left\{-\left[\cosh\left(\frac{q}{m}\right)+\frac{a_{11}\sinh\left(\frac{q}{m}\right)}{qa_{12}}\right]y(0)+y\left(\frac{1}{m}\right)+\frac{c\sinh\left(\frac{q}{m}\right)}{qa_{12}}-\right. \\
\left.-\frac{\sinh\left(\frac{q}{m}\right)}{2qn}f(0,y(0))-\frac{1}{qn}\sum_{k=1}^{r-1}\sinh\left(\frac{q}{m}\left(1-\frac{k}{n}\right)\right)f\left(\frac{k}{n},y\left(\frac{k}{n}\right)\right)\right\}, \text{ if } a_{12}>0, \\
u\left(\frac{i-1}{m}\right)-2\cosh\left(\frac{q}{m}\right)u\left(\frac{i}{m}\right)+u\left(\frac{i+1}{m}\right)-\frac{\sinh\left(\frac{q}{m}\right)}{qm}f_y'\left(\frac{i}{m},y\left(\frac{i}{m}\right)\right)u\left(\frac{i}{m}\right)= \\
=-\left\{\left[\left[y\left(\frac{i-1}{m}\right)-y\left(\frac{i}{m}\right)\right]+\left[y\left(\frac{i+1}{m}\right)-y\left(\frac{i}{m}\right)\right]\right]-4\sinh^2\left(\frac{q}{2m}\right)y\left(\frac{i}{m}\right)-\right. \\
-\frac{1}{qn}\sum_{k=1}^{r}\sinh\left(\frac{q}{m}\frac{k}{r}\right)f\left(\frac{i-1}{m}+\frac{k}{n},y\left(\frac{i-1}{m}+\frac{k}{n}\right)\right)+ \\
\left.+\sum_{k=1}^{r-1}\sinh\left(\frac{q}{m}\left(1-\frac{k}{r}\right)\right)f\left(\frac{i}{m}+\frac{k}{n},y\left(\frac{i}{m}+\frac{k}{n}\right)\right)\right\}, \quad i=1,\ldots,m-1, \\
u\left(\frac{m-1}{m}\right)-\left[\cosh\left(\frac{q}{m}\right)+\frac{b_{21}\sinh\left(\frac{q}{m}\right)}{qb_{22}}\right]u(1)-\frac{\sinh\left(\frac{q}{m}\right)}{2qm}f_y'(1,y(1))u(1)=
\end{cases}
$$

$$
\left.
\begin{aligned}
&= -\left\{ y\left(\frac{m-1}{m}\right) - \left[\cosh\left(\frac{q}{m}\right) + \frac{b_{21}\sinh\left(\frac{q}{m}\right)}{q b_{22}}\right] y(1) + \frac{d\cdot\sinh\left(\frac{q}{m}\right)}{q b_{22}} - \right. \\
&- \frac{1}{qn}\sum_{k=1}^{r-1} \sinh\left(\frac{q}{m}\,\frac{k}{r}\right) f\left(\frac{m-1}{m} + \frac{k}{n}, y\left(\frac{m-1}{m} + \frac{k}{n}\right)\right) - \\
&\left. - \frac{\sinh\left(\frac{q}{m}\right)}{2qn} f(1, y(1)) \right\}, \quad \text{if} \quad b_{22} > 0,
\end{aligned}
\right|
$$

and

$$
\begin{aligned}
(5.16)\quad y\left(\frac{i}{m} + \frac{j}{n}\right) + u\left(\frac{i}{m} + \frac{j}{n}\right) &= \frac{\sinh\left(\frac{q}{m}\left(1 - \frac{j}{r}\right)\right)}{\sinh\left(\frac{q}{m}\right)} \left[y\left(\frac{i}{m}\right) + u\left(\frac{i}{m}\right)\right] + \\
&+ \frac{\sinh\left(\frac{q}{m}\,\frac{k}{r}\right)}{\sinh\left(\frac{q}{m}\right)} \left[y\left(\frac{i+1}{m}\right) + u\left(\frac{i+1}{m}\right)\right] - \\
&- \frac{\sinh\left(\frac{q}{m}\left(1 - \frac{j}{r}\right)\right)}{\sinh\left(\frac{q}{m}\right)} \sum_{k=1}^{j} \frac{1}{qn} \sinh\left(\frac{q}{m}\,\frac{k}{r}\right) f\left(\frac{i}{m} + \frac{k}{n}, y\left(\frac{i}{m} + \frac{k}{n}\right)\right) - \\
&- \frac{\sinh\left(\frac{q}{m}\,\frac{j}{r}\right)}{\sinh\left(\frac{q}{m}\right)} \sum_{k=j+1}^{r} \frac{1}{qn} \sinh\left(\frac{q}{m}\left(1 - \frac{k}{r}\right)\right) f\left(\frac{i}{m} + \frac{k}{n}, y\left(\frac{i}{m} + \frac{k}{n}\right)\right),
\end{aligned}
$$

for $j = 1,\ldots,r-1$, $i = 0,\ldots,m-1$. We get a new y by

$$
y\left(\frac{j}{n}\right) := y\left(\frac{j}{n}\right) + u\left(\frac{j}{n}\right), \quad j = 0,\ldots,n.
$$

The sums in (5.15) and (5.16) can be evaluated by the following algorithm.

$$
(5.17)\quad \left\{
\begin{aligned}
&SU(i,0) = 0;\\
&SU(i,k+1) = SU(i,k) + \sinh\left(\frac{q}{m}\,\frac{k+1}{r}\right) f\left(\frac{i}{m} + \frac{k+1}{n}, y\left(\frac{i}{m} + \frac{k+1}{n}\right)\right);\\
&k = 0,\ldots,r-1,
\end{aligned}
\right.
$$

$$(5.18)\begin{cases} SD(i,r) = 0; \\ SD(i,r-k-1) = SD(i,r-k) + \sinh(\frac{q}{m}\frac{k+1}{r})f(\frac{i+1}{m} - \frac{k+1}{n}, y(\frac{i+1}{m} - \frac{k+1}{n})); \\ k = 0,\ldots,r-2, \end{cases}$$

for $i = 0,\ldots,m-1$. Then we have

$$(5.19)\quad \frac{1}{qn}\sum_{k=1}^{r-1} \sinh(\frac{q}{m}(1 - \frac{k}{r}))f(\frac{k}{n}, y(\frac{k}{n})) = SD(0,1)/(qn),$$

$$(5.20)\quad \frac{1}{qn}\Big[\sum_{k=1}^{r} \sinh(\frac{q}{m}\frac{k}{r})f(\frac{i-1}{m} + \frac{k}{n}, y(\frac{i-1}{m} + \frac{k}{n})) +$$

$$+ \sum_{k=1}^{r-1} \sinh(\frac{q}{m}(1 - \frac{k}{r}))f(\frac{i}{m} + \frac{k}{n}, y(\frac{i}{m} + \frac{k}{n}))\Big] =$$

$$= [SU(i-1,r) + SD(i,1)]/(qn), \quad i = 1,\ldots,m-1,$$

$$(5.21)\quad \frac{1}{qn}\sum_{k=1}^{r-1} \sinh(\frac{q}{m}\frac{k}{r})f(\frac{m-1}{m} + \frac{k}{n}, y(\frac{m-1}{m} + \frac{k}{n})) = SU(m-1,r-1)/(qn),$$

and

$$(5.22)\quad \frac{\sinh(\frac{q}{m}(1 - \frac{j}{r}))}{\sinh(\frac{q}{m})} \sum_{k=1}^{j} \frac{1}{qn} \sinh(\frac{q}{m}\frac{k}{r})f(\frac{i}{m} + \frac{k}{n}, y(\frac{i}{m} + \frac{k}{n})) +$$

$$+ \frac{\sinh(\frac{q}{m}\frac{j}{r})}{\sinh(\frac{q}{m})} \sum_{k=j+1}^{r} \frac{1}{qn} \sinh(\frac{q}{m}(1 - \frac{k}{r}))f(\frac{i}{m} + \frac{k}{n}, y(\frac{i}{m} + \frac{k}{n})) =$$

$$= [\sinh(\frac{q}{m}(1 - \frac{j}{r}))SU(i,j) + \sinh(\frac{q}{m}\frac{j}{r})SD(i,j+1)]/(qn \sinh(\frac{q}{m})),$$

for $j = 1,\ldots,r-1$, $i = 0,\ldots,m-1$.

Newton-like methods for the numerical solution of boundary value problems of the form (5.1) with $q > 0$ have also been given in Aulin [6].

$O(h^4)$-methods for (5.1) can be obtained in exactly the same way as in chapter 4.

6. NEWTON-LIKE METHODS FOR DISCRETE BOUNDARY VALUE PROBLEMS

We define a symmetric discrete boundary value problem by the following equations

$$(6.1)\quad \begin{cases} d_0 y_0 + c_1 y_1 = f_0(y_0), \\ c_j y_{j-1} + d_j y_j + c_{j+1} y_{j+1} = f_j(y_j), \quad j = 1,\dots,n-1, \\ c_n y_{n-1} + d_n y_n = f_n(y_n), \end{cases}$$

where c_j, d_j are real numbers and f_j is continuously differentiable in y_j for $j = 0,\dots,n$. We also assume that $c_j \neq 0$, $j = 1,\dots,n$, since if one $c_j = 0$ the problem splits into two boundary value problems. Let T denote the symmetric tridiagonal matrix of the left hand side of (6.1), and put $f(y) = (f_0(y_0),\dots,f_n(y_n))^T$, $y = (y_0,\dots,y_n)^T$. Then equation (6.1) can be written as

$$(6.2)\quad Ty = f(y).$$

Theorem 6.1

If T is nonsingular then there exists two sequences $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^n$ of real numbers such that $T^{-1} = G = (g_{j,k})_{j,k=0}^n$, where

$$(6.3)\quad g_{j,k} = \begin{cases} b_j a_k, & j \geq k, \\ a_j b_k, & j \leq k. \end{cases}$$

Proof: (See Aulin [10].)

Let $a = (a_0,a_1,\dots,a_n)^T$, $b = (b_0,b_1,\dots,b_n)^T$ then we have

$$(6.4)\quad Tb = (1,0,\dots,0)^T$$

and

$$(6.5)\quad Ta = (0,\dots,0,1/b_n)^T,$$

where $a_0 = 1$. Put

$$d(i,j) = \begin{vmatrix} b_i & a_i \\ b_j & a_j \end{vmatrix}$$

then we also have

$$(6.6) \qquad c_j = \frac{-1}{d(j-1,j)}, \qquad j = 1,\ldots,n,$$

and

$$(6.7) \qquad d_j = \frac{d(j-1,j+1)}{d(j-1,j)d(j,j+1)}, \qquad j = 0,\ldots,n,$$

where $a_{-1} = b_{n+1} = 0$ and $b_{-1} \neq 0$, $a_{n+1} \neq 0$ can be chosen arbitrary (see Aulin [10]).

Since $a_0 = 1$ we can compute a_j, for $j = 1,\ldots,n$, iteratively by (6.5) and also, by the last equation, b_n. With b_n known (6.4) defines an iterative method to compute b_j for $j = n-1, n-2,\ldots,0$. However, since stability problem can occur it might be better to solve the linear system (6.4) first and after that the system (6.5). Then we also have the check for $a_0 = 1$.

We will now give an algorithm for the numerical solution of the two systems (6.4) and (6.5). The algorithm is based on the direct method for tridiagonal systems and (6.6). Let a[0:n], b[0:n+1], c[0:n+1], d[0:n] denote, respectively, the vectors $\{a_j\}$, $\{b_j\}$, $\{c_j\}$, and $\{d_j\}$, then the solution of the two systems (6.4) and (6.5) is given by the following ALGOL program:

```
c[0] := 1; c[n+1] := 0;
for  j := n  step  -1  until  0  do
     b[j] := c[j]/(d[j] - c[j+1] × b[j+1]);
a[0] := 1;
for  j := 1  step   1  until  n  do
begin
     a[j] := - (b[j] × a[j-1] + 1/(b[j-1] × c[j]));
     b[j] := - b[j] × b[j-1];
end;
```

If we compare this algorithm with the iterative algorithm defined by (6.4) and (6.5) we see that the number of operations and overhead (two "for" statements) are exactly the same. For more details see Aulin [10].

Now if T is nonsingular then (6.1) can be written as

$$(6.8)\qquad y_j = \sum_{k=0}^{n} g_{j,k} f_k(y_k), \quad j = 0,\dots,n,$$

where

$$(6.9)\qquad g_{j,k} = \begin{cases} b_j a_k, & j \geq k, \\ a_j b_k, & j \leq k. \end{cases}$$

Equation (5.8) is called the sum equation formulation of (6.1).

We will now derive the determinant formulation of (6.8). Let

$$0 = n_0 < n_1 < \dots < n_{m-1} < n_m = n, \quad m \leq n,$$

be integers. Assume that $a_0 \neq 0$, $b_n \neq 0$, and $d(n_{i-1},n_i) \neq 0$ for $i = 1,\dots,m$. By exactly the same method as used in the continuous case we get the following determinant formulation of (6.8) (see Aulin [10]).

$$(6.10)\qquad \frac{-y_{n_{i-1}}}{d(n_{i-1},n_i)} + \frac{d(n_{i-1},n_{i+1})}{d(n_{i-1},n_i)d(n_i,n_{i+1})} y_{n_i} + \frac{-y_{n_{i+1}}}{d(n_i,n_{i+1})} =$$

$$= \sum_{k=n_{i-1}+1}^{n_{i+1}-1} \Delta_{i,k} f_k(y_k), \quad i = 1,\dots,m-1,$$

where

$$(6.11)\qquad \Delta_{i,k} = \begin{cases} d(n_{i-1},k)/d(n_{i-1},n_i), & k = n_{i-1},\dots,n_i, \\ d(k,n_{i+1})/d(n_i,n_{i+1}), & k = n_i,\dots,n_{i+1}, \end{cases}$$

$$(6.12)\qquad \frac{a_{n_1}}{a_0 d(0,n_1)} y_0 + \frac{-y_{n_1}}{d(0,n_1)} = \sum_{k=0}^{n_1-1} \Delta_{0,k} f_k(y_k),$$

where

$$(6.13)\quad \Delta_{0,k} = d(k,n_1)/d(0,n_1), \quad k = 0,\ldots,n_1,$$

and

$$(6.14)\quad \frac{-y_{n_{m-1}}}{d(n_{m-1},n)} + \frac{b_{n_{m-1}}}{b_n d(n_{m-1},n)} y_n = \sum_{k=n_{m-1}+1}^{n} \Delta_{n,k} f_k(y_k),$$

where

$$(6.15)\quad \Delta_{n,k} = d(n_{m-1},k)/d(n_{m-1},n), \quad k = n_{m-1},\ldots,n.$$

Let T_m denote the matrix of the left hand side of (6.10), (6.12), and (6.14). Then by theorem 3.3 in Aulin [10] T_m is nonsingular if and only if $a_0 \neq 0$, $b_n \neq 0$, and $d(n_{i-1},n_i) \neq 0$ for $i = 1,\ldots,m$. In Aulin [10] it is also shown that $d(k,k+r) \neq 0$ for $0 \leq k < k+r \leq n$ if $c_j > 0$, $j = 1,\ldots,n$, and T is negative definite. The condition that T is negative definite can be replaced by $c_j + c_{j+1} \leq -d_j$, $j = 1,\ldots,n-1$, and T is nonsingular.

To define Newton-like methods for (6.8) we first consider Newton's method for this equation

$$(6.16)\begin{cases} u_j - \sum_{k=0}^{n} g_{j,k} f_k'(y_k) u_k = -\left[y_j - \sum_{k=0}^{n} g_{j,k} f_k(y_k)\right], & j = 0,\ldots,n, \\ y_j := y_j + u_j, & j = 0,\ldots,n. \end{cases}$$

We have

$$\sum_{k=0}^{n} g_{j,k} f_k'(y_k) u_k = \sum_{i=0}^{m-1} \sum_{k=n_i}^{n_i+1} w_k g_{j,k} f_k'(y_k) u_k,$$

where

$$w_k = \begin{cases} \frac{1}{2}, & \text{for} \quad k = n_1, n_2, \ldots, n_{m-1} \\ 1, & \text{otherwise.} \end{cases}$$

One way to approximate the inner sum is by

$$\sum_{k=n_i}^{n_{i+1}} w_k g_{j,k} f'_k(y_k) u_k =$$

$$= \tfrac{1}{2}\left[g_{j,n_i} f'_{n_i}(y_{n_i}) u_{n_i} + g_{j,n_{i+1}} f'_{n_{i+1}}(y_{n_{i+1}}) u_{n_{i+1}} \right] \sum_{k=n_i}^{n_{i+1}} w_k,$$

where

$$\sum_{k=n_i}^{n_{i+1}} w_k = \begin{cases} n_1 - n_0 + \tfrac{1}{2}, & i = 0 \\ n_{i+1} - n_i, & i = 1,\dots,m-2 \\ n_m - n_{m-1} + \tfrac{1}{2}, & i = m-1. \end{cases}$$

Put

$$(6.17)\quad \omega_i = \begin{cases} \tfrac{1}{2}(n_1 - n_0 + \tfrac{1}{2}), & i = 0, \\ \tfrac{1}{2}(n_2 - n_0 + \tfrac{1}{2}), & i = 1, \\ \tfrac{1}{2}(n_{i+1} - n_{i-1}), & i = 2,\dots,m-2, \\ \tfrac{1}{2}(n_m - n_{m-2} + \tfrac{1}{2}), & i = m-1, \\ \tfrac{1}{2}(n_m - n_{m-1} + \tfrac{1}{2}), & i = m. \end{cases}$$

We get the following Newton-like method

$$(6.18)\quad \begin{cases} u_j - \sum_{i=0}^{m} \omega_i g_{j,n_i} f'_{n_i}(y_{n_i}) u_{n_i} = -\left[y_j - \sum_{k=0}^{n} g_{j,k} f_k(y_k) \right], \\ y_j := y_j + u_j, \quad j = 0,\dots,n. \end{cases}$$

This equation corresponds to equation (3.4) in the continuous case. We assume that $a_0 \neq 0$, $b_n \neq 0$, and that $d(n_i, n_{i+1}) \neq 0$ for $i = 0,\dots,m-1$. To obtain an effective method we first apply the determinant formulation to (6.18) for $j \in \{n_0, n_1, \dots, n_m\}$. As in the continuous case we get

$$(6.19)\quad \frac{-u_{n_{i-1}}}{d(n_{i-1},n_i)} + \frac{d(n_{i-1},n_{i+1})}{d(n_{i-1},n_i)d(n_i,n_{i+1})}u_{n_i} + \frac{-u_{n_{i+1}}}{d(n_i,n_{i+1})} -$$

$$- \omega_i f'_{n_i}(y_{n_i})u_{n_i} =$$

$$= -\left\{\frac{-y_{n_{i-1}}}{d(n_{i-1},n_i)} + \frac{d(n_{i-1},n_{i+1})}{d(n_{i-1},n_i)d(n_i,n_{i+1})}y_{n_i} + \frac{-y_{n_{i+1}}}{d(n_i,n_{i+1})} -\right.$$

$$\left. - \sum_{k=n_{i-1}+1}^{n_{i+1}-1} \Delta_{i,k} f_k(y_k)\right\}, \quad i = 1,\dots,m-1,$$

$$(6.20)\quad \frac{a_{n_1}}{a_0 d(0,n_1)}u_0 + \frac{-u_{n_1}}{d(0,n_1)} - \omega_0 f'_0(y_0)u_0 =$$

$$= -\left\{\frac{a_{n_1}}{a_0 d(0,n_1)}y_0 + \frac{-y_{n_1}}{d(0,n_1)} - \sum_{k=0}^{n_1-1} \Delta_{0,k} f_k(y_k)\right\},$$

and

$$(6.21)\quad \frac{-u_{n_{m-1}}}{d(n_{m-1},n)} + \frac{b_{n_{m-1}}}{b_n d(n_{m-1},n)}u_n - \omega_n f'_n(y_n)u_n =$$

$$= -\left\{\frac{-y_{n_{m-1}}}{d(n_{m-1},n)} + \frac{b_{n_{m-1}}}{b_n d(n_{m-1},n)}y_n - \sum_{k=n_{m-1}+1}^{n} \Delta_{m,k} f_k(y_k)\right\}.$$

To compute intermediate values of u_j consider the determinant formulation of equation (6.18) for $n_i < j < n_{i+1}$, we get

$$(6.22)\quad y_j + u_j = \Delta_{i,j}\left[y_{n_i} + u_{n_i}\right] + \Delta_{i+1,j}\left[y_{n_{i+1}} + u_{n_{i+1}}\right] +$$

$$+ d(n_i,n_{i+1})\left[\Delta_{i,j} \sum_{k=n_i+1}^{j} \Delta_{i+1,k} f_k(y_k) + \Delta_{i+1,j} \sum_{k=j+1}^{n_{i+1}} \Delta_{i,k} f_k(y_k)\right],$$

for $j = n_i+1,\dots,n_{i+1}-1$, $i = 0,\dots,m-1$.

To obtain the Newton-like method defined by (6.19), (6.20), (6.21), and (6.22) we have to compute the two vectors $\{a_0,\ldots,a_n\}$ and $\{b_0,\ldots,b_n\}$. This can be done by solving the linear systems (6.4) and (6.5). However in Newton-like methods the original system with n+1 equations was reduced to a system with m+1 equations ($m < n$). Furthermore the sums in Newton-like methods could be evaluated in parallel. By (6.4) and (6.5) we have

$$b_{n_i}\left[c_{n_i+j}a_{n_i+j-1} + d_{n_i+j}a_{n_i+j} + c_{n_i+j+1}a_{n_i+j+1}\right] = 0$$

$$a_{n_i}\left[c_{n_i+j}b_{n_i+j-1} + d_{n_i+j}b_{n_i+j} + c_{n_i+j+1}b_{n_i+j+1}\right] = 0$$

which give

$$c_{n_i+j}d(n_i,n_i+j-1) + d_{n_i+j}d(n_i,n_i+j) + c_{n_i+j+1}d(n_i,b_i+j+1) = 0, \tag{6.23}$$

for $j = 1,\ldots,n_{i+1}-n_i-1$, or if we divide (6.23) by $d(n_i,n_{i+1})$, we get

$$c_{n_i+j}\Delta_{i+1,j-1} + d_{n_i+j}\Delta_{i+1,j} + c_{n_i+j+1}\Delta_{i+1,j+1} = 0, \tag{6.24}$$

for $j = n_i+1,\ldots,n_{i+1}-1$. However we know that

$$\Delta_{i+1,n_i} = \frac{d(n_i,n_i)}{d(n_i,n_{i+1})} = 0, \quad \Delta_{i+1,n_{i+1}} = \frac{d(n_i,n_{i+1})}{d(n_i,n_{i+1})} = 1. \tag{6.25}$$

Hence $\Delta_{i,k}$ can be obtained by solving relatively small tridiagonal systems in parallel. Put

$$T_{i,i+1} = \begin{pmatrix} d_{n_i+1} & c_{n_i+2} & & & 0 \\ c_{n_i+2} & d_{n_i+2} & c_{n_i+3} & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n_{i+1}-2} & d_{n_{i+1}-2} & c_{n_{i+1}-1} \\ 0 & & & c_{n_{i+1}-1} & d_{n_{i+1}-1} \end{pmatrix} \tag{6.26}$$

for $i = 0,1,\ldots,m-1$.

Theorem 6.2

Define the vectors Δ_i^+ and Δ_i^- by

$$(6.27)\begin{cases} \Delta_i^+ = (\Delta_{i+1,n_i+1},\ldots,\Delta_{i+1,n_{i+1}-1})^T, \\ \Delta_i^- = (\Delta_{i,n_i+1},\ldots,\Delta_{i,n_{i+1}-1})^T. \end{cases}$$

Then we have

$$(6.28)\begin{cases} T_{i,i+1}\,\Delta_i^+ = (0,\ldots,0,\ -c_{n_{i+1}})^T, \\ T_{i,i+1}\,\Delta_i^- = (-c_{n_i+1},0,\ldots,0)^T, \end{cases}$$

for $i = 0,\ldots,m-1$, and Δ_i^+, Δ_i^- are unequely defined by (6.28) if $c_j > 0$, $j = 1,\ldots,n$, and T is negative definit, or $c_j + c_{j+1} \le -d_j$, $j = 1,\ldots,n-1$ and T is nonsingular.

Proof: (See Aulin [10] theorem 8.6.)

Now by Theorem 6.2 $\Delta_{i,j}$, $i = 0,1,\ldots,m$, are defined by $2m$ tridiagonal systems which all can be solved in parallel. The elements of the matrix T_m defined by the left hand side of (6.19), (6.20), (6.21) are given by the following theorem.

Theorem 6.3

We have

$$(6.29)\quad \frac{-1}{d(n_i,n_{i+1})} = c_{n_i+1}\Delta_{i+1,n_i+1} = c_{n_{i+1}}\Delta_{i,n_{i+1}-1},$$

for $i = 0,\ldots,m-1$,

$$(6.30)\quad \frac{d(n_{i-1},n_{i+1})}{d(n_{i-1},n_i)d(n_i,n_{i+1})} = c_{n_i}\Delta_{i,n_i-1} + d_{n_i}\Delta_{i,n_i} + c_{n_i+1}\Delta_{i,n_i+1},$$

for $i = 1,\ldots,m-1$ $(\Delta_{i,n_i} = 1)$,

$$(6.31)\qquad \frac{a_{n_1}}{a_0 d(0,n_1)} = d_0 + c_1\Delta_{0,1},$$

$$(6.32)\qquad \frac{b_{n_{m-1}}}{b_n d(n_{m-1},n)} = c_n\Delta_{m,n-1} + d_n.$$

Proof: (See Aulin [10] theorem 8.7.)

If we are not able to do the computations in parallel or if all the 2m systems cannot be solved in parallel at the same time, we can use the following algorithm. This algorithm is based on the algorithm for the solution of the two systems (6.4) and (6.5) and Theorem 6.1. Consider the two systems

$$(6.33)\qquad T_{i,i+1}b = (1,0,\ldots,0)^T$$

and

$$(6.34)\qquad T_{i,i+1}a = (0,\ldots,0,1/b_{n_{i+1}-1})^T,$$

where $a = (a_{n_i+1},\ldots,a_{n_{i+1}-1})^T$ and $b = (b_{n_i+1},\ldots,b_{n_{i+1}-1})^T$.

By the direct method for tridiagonal systems and (6.6) we can solve (6.33) and (6.34) by the following algorithm

$$(6.35)\qquad \begin{cases} w_{n_{i+1}-1} = \dfrac{c_{n_{i+1}-1}}{d_{n_{i+1}-1}}, \\ w_j = \dfrac{c_j}{d_j - c_{j+1}w_{j+1}}, \quad j = n_{i+1}-2,\ldots,n_i+2, \\ w_{n_i+1} = \dfrac{1}{d_{n_i+1} - c_{n_i+2}w_{n_i+2}}, \end{cases}$$

$$(6.36)\qquad \begin{cases} b_{n_i+1} = w_{n_i+1}, \\ b_j = -w_j b_{j-1}, \quad j = n_i+2,\ldots,n_{i+1}-1, \end{cases}$$

and

$$(6.37)\begin{cases} a_{n_i+1} = 1, \\ a_j = \dfrac{b_j}{b_{j-1}} a_{j-1} - \dfrac{1}{b_{j-1}c_j}, \quad j = n_i+2,\ldots,n_{i+1}-1, \end{cases}$$

where $b_j/b_{j-1} = -w_j$.

Clearly we have

$$\Delta_i^- = -c_{n_i+1} b \quad \text{and} \quad \Delta_i^+ = -c_{n_{i+1}} b_{n_{i+1}-1} a.$$

By the direct method for tridiagonal systems we can solve the system

$$T_{i,i+1}\Delta_i^- = (-c_{n_i+1}, 0, \ldots, 0)^T$$

by the following algorithm.

$$(6.38)\begin{cases} w_{n_{i+1}-1} = \dfrac{c_{n_{i+1}-1}}{d_{n_{i+1}-1}}, \\ w_j = \dfrac{c_j}{d_j - c_{j+1}w_{j+1}}, \quad j = n_{i+1}-2,\ldots,n_i+2, \\ w_{n_i+1} = \dfrac{-c_{n_i+1}}{d_{n_i+1} - c_{n_i+2}w_{n_i+2}}, \end{cases}$$

$$(6.39)\begin{cases} \Delta_{i,n_i+1} = w_{n_i+1}, \\ \Delta_{i,j} = -w_j\Delta_{i,j-1}, \quad j = n_i+2,\ldots,n_{i+1}-1. \end{cases}$$

By (6.37) we get

$$(6.40)\begin{cases} \Delta_{i+1,n_i+1} = \dfrac{c_{n_{i+1}}}{c_{n_i+1}} \Delta_{i,n_{i+1}-1}, \\ \Delta_{i+1,j} = -w_j\Delta_{i+1,j-1} + \dfrac{c_{n_{i+1}}\Delta_{i,n_{i+1}-1}}{\Delta_{i,j-1}c_j}, \quad j = n_i+2,\ldots,n_{i+1}-1. \end{cases}$$

We also have

$$(6.41)\quad \Delta_{i,n_{i+1}-1} = w_{n_i+1}(-w_{n_i+2})\ldots(-w_{n_{i+1}-1}).$$

In pseudo ALGOL the algorithm for the evaluation of the vectors Δ_i^+, Δ_i^-, $i = 0,\ldots,m-1$, can be written in the following form.

```
for i := 0 step 1 until m-1 do
begin
   W := 1;
   Δ⁻[i,n_i] := 1; Δ⁻[i,n_{i+1}] := 0;
   for j := n_{i+1}-1 step -1 until n_i+1 do
   begin
      Δ⁻[i,j] := c[j]/(d[j] - c[j+1] × Δ⁻[i,j+1]);
      W := - W × Δ⁻[i,j]
   end;
   W = W × c[n_{i+1}];
   Δ⁻[i,n_i+1] := - Δ⁻[i,n_i+1];
   Δ⁺[i,n_i+1] := W/c[n_i+1];
   for j := n_i+2 step 1 until n_{i+1}-1 do
   begin
      Δ⁺[i,j] := W/(Δ⁻[i,j-1] × c[j]) - Δ⁻[i,j] × Δ⁺[i,j-1];
      Δ⁻[i,j] := - Δ⁻[i,j] × Δ⁻[i,j-1]
   end;
end i ;
```

7. CONVERGENCE THEOREMS

Let X be a Banach space and $S(y_0,t) = \{ y \in X \mid \|y - y_0\| < t \}$, $\overline{S}(y_0,t)$ its closure and let $L(X)$ be the Banach space of bounded linear operators on X. First we need a lemma.

Lemma 7.1

Let X be a Banach space and let $S,T \in L(X)$. Assume there exists an operator $S^{-1} \in L(X)$ satisfying $\|S^{-1}\| \cdot \|S - T\| < 1$. Then there exists an operator $T^{-1} \in L(X)$ such that

$$(7.1) \qquad \|T^{-1}\| < \frac{\|S^{-1}\|}{1 - \|S^{-1}\| \cdot \|S - T\|} .$$

Proof: (See Anselone [1]).

The following convergence theorem for Newton-like methods is a slight modification of theorem 12.6.4 in Ortega and Rheinboldt [14].

Theorem 7.2

Suppose that $P: X \longrightarrow X$ is continuously Fréchet differentiable on the convex set $D_0 \subset X$ and

$$(7.2) \qquad \|P'(y) - P'(z)\| \leq \gamma \|y - z\|, \quad y,z \in D_0,$$

$$(7.3) \qquad \|A(y) - A(z)\| \leq \gamma \|y - z\|, \quad y,z \in D_0,$$

where $A: D_0 \longrightarrow L(X)$. Let $y_0 \in D_0$ be such that $A(y_0)$ is nonsingular and

$$(7.4) \qquad \|A(y_0)^{-1}\| \leq \beta ,$$

$$(7.5) \qquad \|A(y_0)^{-1} P(y_0)\| \leq \eta .$$

Further assume that

$$(7.6) \qquad \|P'(y) - A(y)\| \leq C, \quad y \in D_0,$$

and that $\beta C < 1$ and

$$\alpha = \beta\gamma\eta/(1 - \beta C)^2 < \tfrac{1}{2}.$$

Set

$$(7.7)\quad t^{*} = \frac{1 - \sqrt{1 - 2\alpha}}{\alpha(1 - \beta C)}\eta,$$

$$(7.8)\quad t^{**} = \frac{1 + \sqrt{1 - 2\alpha}}{\alpha(1 - \beta C)}\eta.$$

If $\overline{S}(y_0, t^{*}) \subset D_0$, then the sequence $\{y_n\}$ defined by

$$(7.9)\quad y_{n+1} = y_n - A(y_n)^{-1}P(y_n), \quad n = 0,1,\ldots,$$

remains in $S(y_0, t^{*})$ and converge to a solution y^{*} of $P(y) = 0$ which is unique in $D_0 \cap S(y_0, t^{**})$.

Proof: For $y \in S(y_0, t^{*})$, we have

$$\|A(y) - A(y_0)\| \le \gamma\|y - y_0\| < \gamma t^{*} \le \frac{1 - \beta C}{\beta}(1 - \sqrt{1-2\alpha}) < \frac{1}{\beta}$$

$$\Rightarrow \quad \|A(y_0)^{-1}\| \cdot \|A(y) - A(y_0)\| < 1.$$

Hence, by Lemma 7.1 $A(y)$ is nonsingular and

$$\|A(y)^{-1}\| \le \frac{\beta}{1 - \beta\gamma\|y - y_0\|}, \quad y \in S(y_0, t^{*}).$$

Therefore, the mapping $Gy = y - A(y)^{-1}Py$ is defined for $y \in S(y_0, t^{*})$ and if $y, Gy \in S(y_0, t^{*})$, then

$$\|G^2y - Gy\| = \|A(Gy)^{-1}P(Gy)\| \le$$

$$\le \frac{\beta}{1 - \beta\gamma\|Gy - y_0\|}\left[\|P(Gy) - P(y) - P'(y)(Gy - y)\| + \right.$$

$$\left. + \|[P'(y) - A(y)](Gy - y)\|\right] \le$$

$$\le \frac{\beta}{1 - \beta\gamma\|Gy - y_0\|}\left[\frac{\gamma}{2}\|Gy - y\|^2 + C\|Gy - y\|\right].$$

Therefore we have, assuming that $y, Gy \in S(y_0, t^{*})$,

$$\|G^2y - Gy\| \le h(\|Gy - y\|, \|Gy - y_0\|),$$

where

$$h(s,t) = \frac{\frac{\beta\gamma}{2} s^2 + \beta C s}{1 - \beta\gamma t}.$$

Consider the difference equation

(7.10) $t_{n+1} - t_n = h(t_n - t_{n-1}, t_n)$, $t_0 = 0$, $t_1 = \eta$, $n = 1,\ldots$.

It is easy to show that

$$g(t) = t + \frac{\frac{\beta\gamma}{2} t^2 - (1 - \beta C)t + \eta}{1 - \beta\gamma t}$$

is a "first integral" of (7.10), and that the solution of (7.10) satisfies $\lim_{n \to \infty} t_n = t^*$ and $t_n < t^*$, $n \geq 0$, where t^* is given by (7.7).

For the rest of the proof, see Ortega and Rheinboldt [14] section 12.6.

In this chapter we will only consider the boundary value problem

$$(7.11)\quad \begin{cases} (py')' = f(y(t)), & 0 < t < 1, \\ y(0) = c, \quad y(1) = d, \end{cases}$$

where $p > 0$ on the interval $[0,1]$, $p \in C^1[0,1]$, $f \in C^2(R)$. Moreover we will only consider the discretization of this problem given by (4.11) on a equidistant grid $G_n = \left\{ \frac{i}{n} \mid i = 0,\ldots,n \right\}$. Newton-like methods for this problem are defined by (4.14) and (4.15). It will be easy to generalize the convergence proofs of this chapter to the boundary value problem (2.1), both for $O(h^2)$ and $O(h^4)$ discretizations and for general boundary conditions. Convergence theorems for Newton-like methods for the numerical solution of the following problems are given in Aulin [4]:

$$\begin{cases} y'' = f(y) \\ y(0) = a, \quad y(1) = b, \text{ trapezoidal rule,} \end{cases}$$

$$\begin{cases} y'' = f(y) \\ y(0) = a, \quad y(1) = b, \quad O(h^4) \text{ approximation,} \end{cases}$$

$$\begin{cases} y'' = f(y) \\ a_{11}y(0) - a_{12}y'(0) = a \\ b_{21}y(1) + b_{22}y'(1) = b, \quad \text{trapezoidal rule.} \end{cases}$$

Let m,n, and r be positive integers and $n = rm$. Then m and n define two grids, G_m and G_n, on $[0,1]$ with equidistant points. The discretization of the boundary value problem (7.11) is given by (4.11) (with $a_{12} = b_{22} = 0$). Thus we will consider the following discrete equations

$$(7.12)\begin{cases} \dfrac{-y(\frac{i-1}{n})}{D(\frac{i-1}{n},\frac{i}{n})} + \left[\dfrac{1}{D(\frac{i-1}{n},\frac{i}{n})} + \dfrac{1}{D(\frac{i}{n},\frac{i+1}{n})}\right] y(\frac{i}{n}) + \dfrac{-y(\frac{i+1}{n})}{D(\frac{i}{n},\frac{i+1}{n})} = \dfrac{1}{n} f(y(\frac{i}{n})), \\ y(0) = c, \quad y(1) = d, \end{cases}$$

for $i = 1,\ldots,n-1$. Newton-like methods for (7.12) is defined by (4.14) and (4.15) with $a_{12} = b_{22} = 0$. Hence we have

$$(7.13)\begin{cases} \dfrac{-u_s(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\dfrac{1}{D(\frac{i-1}{m},\frac{i}{m})} + \dfrac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right] u_s(\frac{i}{m}) + \dfrac{-u_s(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} - \\ - \dfrac{1}{m} f'(y_s(\frac{i}{m}))u_s(\frac{i}{m}) = \\ = -\left\{ \dfrac{-y_s(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\dfrac{1}{D(\frac{i-1}{m},\frac{i}{m})} + \dfrac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right] y_s(\frac{i}{m}) + \dfrac{-y_s(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} - \right. \\ \left. - \sum_{k=r(i-1)+1}^{r(i+1)-1} \dfrac{1}{n} \Delta_i(\frac{k}{n}) f(y(\frac{k}{n})) \right\}, \quad i = 1,\ldots,m-1, \\ u_s(0) = u_s(1) = 0, \end{cases}$$

$$(7.14)\quad y_{s+1}\left(\frac{i}{m}\right) = y_s\left(\frac{i}{m}\right) + u_s\left(\frac{i}{m}\right), \qquad i = 0,\ldots,m,$$

$$(7.15)\quad y_{s+1}\left(\frac{j}{n}\right) = \frac{D\left(\frac{j}{n},\frac{i+1}{n}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)}\, y_{s+1}\left(\frac{i}{m}\right) + \frac{D\left(\frac{i}{m},\frac{j}{n}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)}\, y_{s+1}\left(\frac{i+1}{m}\right) +$$

$$+ \frac{D\left(\frac{j}{n},\frac{i+1}{m}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)} \sum_{k=ri+1}^{j} \frac{1}{n}\, D\left(\frac{i}{m},\frac{k}{n}\right) f\left(y_s\left(\frac{k}{n}\right)\right) +$$

$$+ \frac{D\left(\frac{i}{m},\frac{j}{n}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)} \sum_{k=j+1}^{r(i+1)} \frac{1}{n}\, D\left(\frac{k}{n},\frac{i+1}{m}\right) f\left(y_s\left(\frac{k}{n}\right)\right),$$

where $y_{s+1}\left(\frac{j}{n}\right) = y_s\left(\frac{j}{n}\right) + u_s\left(\frac{j}{n}\right)$, for $j = ri+1,\ldots,r(i+1)-1$, $i = 0,\ldots$

$\ldots,m-1$.

The sum equation form of (7.13), (7.14), and (7.15) is

$$(7.16)\quad u_s\left(\frac{j}{n}\right) - \sum_{i=1}^{m-1} \frac{1}{m}\, G\left(\frac{j}{n},\frac{i}{m}\right) f'\left(y_s\left(\frac{i}{m}\right)\right) u_s\left(\frac{i}{m}\right) =$$

$$= -\left\{ y_s\left(\frac{j}{n}\right) - \sum_{k=1}^{n-1} \frac{1}{n}\, G\left(\frac{j}{n},\frac{k}{n}\right) f\left(y_s\left(\frac{k}{n}\right)\right) - \left[1 - \frac{P\left(\frac{j}{n}\right)}{P(1)}\right] c - \frac{P\left(\frac{j}{n}\right)}{P(1)}\, d \right\},$$

for $j = 1,\ldots,n-1$, where

$$(7.17)\quad G\left(\frac{j}{n},\frac{k}{n}\right) = \begin{cases} b\left(\frac{j}{n}\right) a\left(\frac{k}{n}\right), & j \geq k, \\ a\left(\frac{j}{n}\right) b\left(\frac{k}{n}\right), & j \leq k, \end{cases}$$

and

$$(7.18)\quad \begin{cases} a\left(\frac{j}{n}\right) = \dfrac{P\left(\frac{j}{n}\right)}{p(1)P(1)}, & j = 0,\ldots,n, \\ b\left(\frac{j}{n}\right) = p(1)\left[P\left(\frac{j}{n}\right) - P(1)\right], & j = 0,\ldots,n, \\ P(x) = \displaystyle\int_0^x dt/p(t), & 0 \leq x \leq 1. \end{cases}$$

Since G has the form ab we can cancel $p(1)$. Thus for the rest of this chapter we put

$$(7.19)\begin{cases} a(\frac{j}{n}) = \dfrac{P(\frac{j}{n})}{P(1)} = \dfrac{D(0,\frac{j}{n})}{D(0,1)} \\ b(\frac{j}{n}) = P(\frac{j}{n}) - P(1) = D(\frac{j}{n},1) \end{cases}$$

for $j = 0,\ldots,n$. Moreover, we assume that $P(x)$ can be evaluated exactly.

We will first consider the two cases $m = 1$ and $m = n$. If $m = 1$ then by (7.15) we have

$$(7.20)\quad y_{s+1}(\tfrac{j}{n}) = \frac{D(\frac{j}{n},1)}{D(0,1)}\,c + \frac{D(0,\frac{j}{n})}{D(0,1)}\,d + D(\tfrac{j}{n},1)\sum_{k=1}^{j}\frac{1}{n}\frac{D(0,\frac{k}{n})}{D(0,1)}f(y_s(\tfrac{k}{n})) +$$

$$+\frac{D(0,\frac{j}{n})}{D(0,1)}\sum_{k=j+1}^{n}\frac{1}{n}D(\tfrac{k}{n},1)f(y_s(\tfrac{k}{n}))$$

for $j = 1,\ldots,n-1$. By (7.18) and (7.19) we can write (7.20) in the following form

$$(7.21)\quad y_{s+1}(\tfrac{j}{n}) = \sum_{k=1}^{n-1}\frac{1}{n}G(\tfrac{j}{n},\tfrac{k}{n})f(y_s(\tfrac{k}{n})) + \frac{cb(\frac{j}{n})}{B_1(b)} + \frac{da(\frac{j}{n})}{B_2(a)},\quad j = 1,\ldots,n-1.$$

Hence the determinant formulation of (7.20) is

$$(7.22)\quad \frac{-y_{s+1}(\frac{j-1}{n})}{D(\frac{j-1}{n},\frac{j}{n})} + \Big[\frac{1}{D(\frac{j-1}{n},\frac{j}{n})} + \frac{1}{D(\frac{j}{n},\frac{j+1}{n})}\Big]y_{s+1}(\tfrac{j}{n}) + \frac{-y_{s+1}(\frac{j+1}{n})}{D(\frac{j}{n},\frac{j+1}{n})} =$$

$$= \frac{1}{n}f(y_s(\tfrac{j}{n})),\quad j = 1,\ldots,n-1.$$

Let

$$(7.23)\quad T_n = \begin{bmatrix} \frac{1}{D(0,\frac{1}{n})} + \frac{1}{D(\frac{1}{n},\frac{2}{n})} & \frac{-1}{D(\frac{1}{n},\frac{2}{n})} & & & 0 \\ \frac{-1}{D(\frac{1}{n},\frac{2}{n})} & \frac{1}{D(\frac{1}{n},\frac{2}{n})} + \frac{1}{D(\frac{2}{n},\frac{3}{n})} & \frac{-1}{D(\frac{2}{n},\frac{3}{n})} & & \\ & \ddots & \ddots & \ddots & \\ & \frac{-1}{D(\frac{n-3}{n},\frac{n-2}{n})} & \frac{1}{D(\frac{n-3}{n},\frac{n-2}{n})} + \frac{1}{D(\frac{n-2}{n},\frac{n-1}{n})} & \frac{-1}{D(\frac{n-2}{n},\frac{n-1}{n})} & \\ 0 & & \frac{-1}{D(\frac{n-2}{n},\frac{n-1}{n})} & \frac{1}{D(\frac{n-2}{n}\frac{n-1}{n})} + \frac{1}{D(\frac{n-1}{n},1)} & \end{bmatrix}$$

and

$$y_{01} = \begin{bmatrix} \frac{c}{D(0,\frac{1}{n})} \\ 0 \\ \vdots \\ 0 \\ \frac{d}{D(\frac{n-1}{n},1)} \end{bmatrix}, \quad y_s = \begin{bmatrix} y_s(\frac{1}{n}) \\ y_s(\frac{2}{n}) \\ \vdots \\ y_s(\frac{n-1}{n}) \end{bmatrix}, \quad u_s = \begin{bmatrix} u_s(\frac{1}{n}) \\ u_s(\frac{2}{n}) \\ \vdots \\ u_s(\frac{n-1}{n}) \end{bmatrix}$$

$$f(y_s) = \begin{bmatrix} f(y_s(\frac{1}{n})) \\ f(y_s(\frac{2}{n})) \\ \vdots \\ f(y_s(\frac{n-1}{n})) \end{bmatrix}$$

With $y_{s+1} = y_s + u_s$ we can write (7.22) in matrix notation as

$$(7.24)\quad T_n u_s = -\left(T_n y_s - y_{01} - \frac{1}{n} f(y_s)\right).$$

Thus (7.24) defines the method of successive approximations for the system

$$(7.25)\quad T_n y - y_{01} = \frac{1}{n} f(y), \qquad y = (y(\tfrac{1}{n}),\ldots,y(\tfrac{n-1}{n}))^T.$$

If $m = n$ then (7.13) defines Newton's method for (7.25). In matrix notation Newton's method can be written in the following form

$$(7.26)\quad \begin{cases} [T_n - \frac{1}{n}D(f'(y_s))]u_s = -[T_n y_s - y_{01} - \frac{1}{n} f(y_s)], \\ y_{s+1} = y_s + u_s, \end{cases}$$

where $D(f'(y_s))$ is a diagonal matrix with diagonal elements $f'(y_s(\frac{j}{n}))$, $j = 1,\ldots,n-1$. Thus the method of successive approximations defined by (7.24) can be considered as a Newton-like method. In Theorem 7.2 we have $A(y_s) = T_n$, $P'(y_s) = T_n - \frac{1}{n} D(f'(y_s))$, and

$$C = \| T_n - \tfrac{1}{n} D(f'(y_s)) - T_n \| = \tfrac{1}{n} \| D(f'(y_s)) \| = \tfrac{1}{n} \| f'(y_s) \|,$$

where $\| f'(y_s) \| = \max_j |f'(y_s(j/n))|$, if the norm is <u>monotonic</u> (or <u>absolute</u>, see Ortega [13] p. 52). We also have

$$\beta = \| T_n^{-1} \|$$

and we want to have $\beta C < 1$. Thus

$$\| f'(y_s) \| < \frac{n}{\| T^{-1} \|}, \quad \text{for all } s,$$

is one of our conditions for the convergence of (7.24) when Theorem 7.2 is applied. (It is easy to see that $\| T^{-1} \| = O(n)$.)

We will now assume that $1 < m < n$. Consider the determinant formulation of (7.16) on the grid G_n. We get

$$(7.27)\quad [T_n - \tfrac{1}{m} \tilde{D}(f'(y_s))]u_s = -[T_n y_s - y_{01} - \tfrac{1}{n} f(y_s)],$$

where $\tilde{D}(f'(y_s)) = \operatorname{diag}(d_1,\ldots,d_{n-1})$ and

$$(7.28)\quad d_k = \begin{cases} \frac{1}{m} f'(y_s(\frac{i}{m})), & \text{for } k = ri, \\ 0, & \text{for } k \neq ri, \quad i = 1,\ldots,m-1. \end{cases}$$

Thus if we apply Theorem 7.2, to the Newton-like method defined by (7.13), (7.14), and (7.15), we get almost the same estimate of the

constant C as for the method of successive approximations. This shows the limitation of Theorem 7.2.

Let

$$K\varphi = \int_0^1 K(x,t,\varphi(t))dt, \quad 0 \le x \le 1,$$

where $K(x,t,\varphi)$ is continuous in x and t, continuously differentiable in φ, and let $K_n\varphi$ be a suitable approximation of $K\varphi$ by a quadrature formula. Then it can be shown that

$$(7.29) \quad \|(K_n'(\varphi) - K'(\varphi))K_n'(\varphi)\| \longrightarrow 0, \quad \|(K_n'(\varphi) - K'(\varphi))K'(\varphi)\| \longrightarrow 0$$

when $n \longrightarrow \infty$, while

$$\|K_n'(\varphi) - K'(\varphi)\| \longrightarrow 2\|K'\| \quad \text{when} \quad n \longrightarrow \infty$$

(see Anselone [1]).

We will now give a convergence theorem for the Newton-like method defined by (7.13), (7.14), and (7.15). This theorem make use of the property (7.29).

Let $T_m - \frac{1}{m} D(f'(y_s))$ denote the $(m-1)\times(m-1)$ matrix of the left hand side of (7.13). y_0 will in general denote the vector $(y_0(\frac{1}{n}),\ldots,y_0(\frac{n-1}{n}))^T \in R^{n-1}$ but in the matrix $T_m - \frac{1}{m} D(f'(y_0))$ we will consider y_0 as a vector in R^{m-1} with elements $y_0(\frac{1}{m}),\ldots,y_0(\frac{m-1}{m})$. The norm $\|\cdot\|$ is the max-norm in R^{n-1}, and $\|\cdot\|_m$ is the max-norm in R^{m-1}. We will also use $\|\cdot\|$ to denote the uniform norm in $C[0,1]$.

Theorem 7.3

Suppose that f is twice continuously differentiable. Let the vector $y_0 \in R^{n-1}$ be such that $T_m - \frac{1}{m} D(f'(y_0))$ is nonsingular and

$$(7.30) \quad \|[T_m - \frac{1}{m} D(f'(y_0))]^{-1}\|_m \le m\beta,$$

where $\beta > 0$ is a fix constant,

$$(7.31) \quad \|u_0\| < \eta, \quad \|u_1\| < \eta d, \quad d < 1.$$

Set $t = \eta/(1-d)$, $S = S(y_0,t) = \{y \in R^{n-1} \mid \|y - y_0\| < t\}$.

Let $I_t(y_0)$ be the convex hull of the intervals $\left[y_0(\frac{k}{n})-t, y_0(\frac{k}{n})+t\right]$, $k = 1,\dots,n-1$. Set

$$\|f\| = \max_{x \in I_t(y_0)} |f(x)|, \quad \|f'\| = \max_{x \in I_t(y_0)} |f'(x)|,$$

$$\|f''\| = \max_{x \in I_t(y_0)} |f''(x)|.$$

Assume that

(7.32) $\beta\|f''\|\eta/(1-d) < 1.$

Set

(7.33) $\rho = \beta/(1 - \frac{\beta\|f''\|\eta}{1-d})$.

If

(7.34) $$\rho\Big\{\|f''\|\eta + \frac{2\|1/p\|}{dm}\|f'\|\,\|f''\|\eta +$$

$$+\Big[\frac{2\|1/p\|}{m}\Big((1+\frac{2}{m})\|f\|\cdot\|f''\| + \Big|\frac{d-c}{D(0,1)}\Big|\|f''\| + \|f'\|^2\Big) +$$

$$+\frac{M}{6m^2}(1+\frac{1}{r^2})\|f'\|\Big]\frac{1}{d} + \frac{4\|1/p\|}{m}(1+\frac{1}{m})\|f'\|^2\frac{1}{d^2}\Big\} +$$

$$+\frac{\|1/p\|}{m^2}\|f'\|\frac{1}{d} < 1$$

where

$$|\Delta_i''(\tfrac{i}{m}+t)| \le M, \quad -\frac{1}{m} < t < 0, \ 0 < t < \frac{1}{m}, \ i = 1,\dots,m-1,$$

$$\|1/p\| = 1/\Big[\min_{0\le t\le 1} p(t)\Big],$$

then the sequence y_s defined by (7.13), (7.14), and (7.15) remains in S and converge to a $y^* \in \overline{S}$ and y^* is the solution of the difference equation (7.12) which is unique in $\overline{S}$. Moreover,

(7.35) $\|y^* - y_s\| \le \eta d^s/(1-d)$.

Proof: For $y \in S(y_0,t)$ we have

$$\|[T_m - \tfrac{1}{m} D(f'(y_0))]^{-1}\|_m \|[T_m - \tfrac{1}{m} D(f'(y))] -$$

$$- [T_m - \tfrac{1}{m} D(f'(y_0))]\|_m \le m\beta \tfrac{1}{m} \|D(f'(y_0) - f'(y))\|_m \le$$

$$\le \beta \|f''\| \cdot \|y_0 - y\| \le \beta \|f''\| \eta/(1-d) < 1$$

by (7.32). Hence, by Lemma 7.1 $T_m - \frac{1}{m} D(f'(y))$ is nonsingular and

(7.36) $$\|[T_m - \tfrac{1}{m} D(f'(y))]^{-1}\|_m \le m\beta/(1 - \frac{\beta\|f''\|\eta}{1-d}) = m\varrho, \quad y \in S(y_0,t).$$

By (7.31) $y_1, y_2 \in S(y_0,t)$ and u_2 is well defined. If $y_0,\dots,y_{s+2} \in S(y_0,t)$ then u_{s+2} is defined and (7.13) gives

(7.37) $$\frac{-u_{s+2}(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\frac{1}{D(\frac{i-1}{m},\frac{i}{m})} \quad \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right] u_{s+2}(\tfrac{i}{m}) + \frac{-u_{s+2}(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -$$

$$- \tfrac{1}{m} f'(y_{s+2}(\tfrac{i}{m})) u_{s+2}(\tfrac{i}{m}) =$$

$$= -\left\{\frac{-y_{s+2}(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\frac{1}{D(\frac{i-1}{m},\frac{i}{m})} + \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right] y_{s+2}(\tfrac{i}{m}) + \frac{-y_{s+2}(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -\right.$$

$$\left. - \frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i(\tfrac{i}{m} + \tfrac{k}{n}) f(y_{s+2}(\tfrac{i}{m} + \tfrac{k}{n}))\right\} +$$

$$+\left\{\frac{-u_{s+1}(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\frac{1}{D(\frac{i-1}{m},\frac{i}{m})} + \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right] u_{s+1}(\tfrac{i}{m}) + \frac{-u_{s+1}(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -\right.$$

$$\left. - \tfrac{1}{m} f'(y_{s+1}(\tfrac{i}{m})) u_{s+1}(\tfrac{i}{m})\right\} +$$

$$+\left\{\frac{-y_{s+1}(\frac{i-1}{m})}{D(\frac{i-1}{m},\frac{i}{m})} + \left[\frac{1}{D(\frac{i-1}{m},\frac{i}{m})} + \frac{1}{D(\frac{i}{m},\frac{i+1}{m})}\right] y_{s+1}(\tfrac{i}{m}) + \frac{-y_{s+1}(\frac{i+1}{m})}{D(\frac{i}{m},\frac{i+1}{m})} -\right.$$

$$\left. - \frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i(\tfrac{i}{m} + \tfrac{k}{n}) f(y_{s+1}(\tfrac{i}{m} + \tfrac{k}{n}))\right\} =$$

$$= \frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right)\left[f\left(y_{s+2}\left(\frac{i}{m} + \frac{k}{n}\right)\right) - f\left(y_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right)\right)\right] -$$

$$- \frac{1}{m} f'\left(y_{s+1}\left(\frac{i}{m}\right)\right) u_{s+1}\left(\frac{i}{m}\right) =$$

$$= \frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right)\left[f\left(y_{s+2}\left(\frac{i}{m} + \frac{k}{n}\right)\right) - f\left(y_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right)\right) -\right.$$

$$\left. - f'\left(y_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right)\right) u_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right)\right] +$$

$$+ \left\{\frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right)\right) u_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right) - \frac{1}{m} f'\left(y_{s+1}\left(\frac{i}{m}\right)\right) u_{s+1}\left(\frac{i}{m}\right)\right\} =$$

$$= S_1 + S_2.$$

Since $0 \leq \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right) \leq 1$, we have

$$(7.38) \quad |S_1| \leq \frac{2}{m} \|f''\| \frac{\|u_{s+1}\|^2}{2} = \frac{1}{m} \|f''\| \cdot \|u_{s+1}\|^2.$$

We write S_2 in the folowing form

$$S_2 = \frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right)\left[f'\left(y_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right)\right) - f'\left(y_{s+1}\left(\frac{i}{m}\right)\right)\right] u_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right) +$$

$$+ \left\{\frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right) f'\left(y_{s+1}\left(\frac{i}{m}\right)\right) u_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right) -\right.$$

$$\left. - \frac{1}{m} f'\left(y_{s+1}\left(\frac{i}{m}\right)\right) u_{s+1}\left(\frac{i}{m}\right)\right\} = S_3 + S_4.$$

Now S_4 can be written as

$$S_4 = \frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right) f'\left(y_{s+1}\left(\frac{i}{m}\right)\right)\left[u_{s+1}\left(\frac{i}{m} + \frac{k}{n}\right) - u_{s+1}\left(\frac{i}{m}\right)\right] +$$

$$+ \left\{\left[\frac{1}{n} \sum_{k=1-r}^{r-1} \Delta_i\left(\frac{i}{m} + \frac{k}{n}\right) - \frac{1}{m}\right] f'\left(y_{s+1}\left(\frac{i}{m}\right)\right) u_{s+1}\left(\frac{i}{m}\right)\right\} =$$

$$= S_5 + S_6.$$

By the remainder of the trapezoidal rule we have

$$\left|\frac{1}{n}\sum_{k=1-r}^{r-1}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)-\frac{1}{m}\right| \le \left|\frac{1}{n}\sum_{k=1-r}^{r-1}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)-\int_{-1/m}^{1/m}\Delta_i\left(\frac{i}{m}+t\right)dt\right| +$$

$$+\left|\int_{-1/m}^{1/m}\Delta_i\left(\frac{i}{m}+t\right)dt-\frac{1}{m}\right| \le \frac{1}{m}\,\frac{1}{12n^2}2M+\frac{1}{12m^3}\,2M=\frac{M}{6m^3}\left(1+\frac{1}{r^2}\right).$$

Thus

$$(7.39)\quad |S_6| \le \frac{M}{6m^3}\left(1+\frac{1}{r^2}\right)\|f'\|\cdot\|u_{s+1}\|.$$

We will now consider S_3. By (7.15) we have

$$(7.40)\quad y_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)-y_{s+1}\left(\frac{i}{m}\right)=\Delta_{i+1}\left(\frac{i}{m}+\frac{k}{n}\right)\left[y_{s+1}\left(\frac{i+1}{m}\right)-y_{s+1}\left(\frac{i}{m}\right)\right]+$$

$$+\frac{D\left(\frac{i}{m},\frac{i+1}{m}\right)}{m}\left\{\frac{1}{r}\sum_{j=1}^{k}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)\Delta_{i+1}\left(\frac{i}{m}+\frac{j}{n}\right)f\left(y_s\left(\frac{i}{m}+\frac{j}{n}\right)\right)+\right.$$

$$\left.+\frac{1}{r}\sum_{j=k+1}^{r-1}\Delta_{i+1}\left(\frac{i}{m}+\frac{k}{n}\right)\Delta_i\left(\frac{i}{m}+\frac{j}{n}\right)f\left(y_s\left(\frac{i}{m}+\frac{j}{n}\right)\right)\right\}$$

for $k = 1,\ldots,r-1$. By (7.16) we have

$$(7.41)\quad y_{s+1}\left(\frac{i}{m}\right)=\frac{1}{m}\sum_{j=1}^{m-1}G\left(\frac{i}{m},\frac{j}{m}\right)f'\left(y_s\left(\frac{j}{m}\right)\right)u_s\left(\frac{j}{m}\right)+\frac{1}{n}\sum_{k=1}^{n-1}G\left(\frac{i}{m},\frac{k}{n}\right)f\left(y_s\left(\frac{k}{n}\right)\right)+$$

$$+c+\frac{D\left(0,\frac{i}{m}\right)}{D(0,1)}[d-c],$$

for $i = 0,\ldots,m$. Let f_j be real numbers, then we have

$$(7.42)\quad \frac{1}{m}\sum_{j=1}^{m-1}\left[G\left(\frac{i+1}{m},\frac{j}{m}\right)-G\left(\frac{i}{m},\frac{j}{m}\right)\right]f_j=$$

$$=D\left(\frac{i}{m},\frac{i+1}{m}\right)\left\{\frac{-1}{m}\sum_{j=1}^{i}\frac{D\left(0,\frac{j}{n}\right)}{D(0,1)}f_j+\frac{1}{m}\sum_{j=i+1}^{m-1}\frac{D\left(\frac{j}{m},1\right)}{D(0,1)}f_j\right\}$$

for $i = 0,\ldots,m-1$ ($\sum_{j=1}^{0} = \sum_{j=m}^{m-1} = 0$). The first difference of the

other sum in (7.41) is

$$(7.43)\quad \frac{1}{n}\sum_{k=1}^{n-1}\left[G\left(\frac{i+1}{m},\frac{k}{n}\right)-G\left(\frac{i}{m},\frac{k}{n}\right)\right]f_k =$$

$$= D\left(\frac{i}{m},\frac{i+1}{m}\right)\left\{\frac{-1}{n}\sum_{k=1}^{r(i+1)}\frac{D\left(0,\frac{k}{n}\right)}{D(0,1)}f_k+\frac{1}{n}\sum_{k=ri+1}^{r(i+1)}\frac{D\left(\frac{i}{m},\frac{k}{n}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)}f_k+\right.$$

$$\left.+\frac{1}{n}\sum_{k=r(i+1)+1}^{n-1}\frac{D\left(\frac{k}{n},1\right)}{D(0,1)}f_k\right\}=$$

$$= D\left(\frac{i}{m},\frac{i+1}{m}\right)\left\{\frac{-1}{n}\sum_{k=1}^{ri}\frac{D\left(0,\frac{k}{n}\right)}{D(0,1)}f_k+\frac{1}{n}\sum_{k=ri+1}^{r(i+1)}\frac{D\left(\frac{k}{n},\frac{i+1}{m}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)}f_k+\right.$$

$$\left.+\frac{1}{n}\sum_{k=ri+1}^{n-1}\frac{D\left(\frac{k}{n},1\right)}{D(0,1)}f_k\right\},$$

for $i = 0,\ldots,m-1$.

By (7.41) (7.42), and (7.43) we now have

$$(7.44)\quad \left|y_{s+1}\left(\frac{i+1}{m}\right)-y_{s+1}\left(\frac{i}{m}\right)\right| \le \left|D\left(\frac{i}{m},\frac{i+1}{m}\right)\right|\left[\|f'\|\cdot\|u_s\|+\right.$$

$$\left.+\left(1+\frac{1}{m}\right)\|f\|+\left|\frac{d-c}{D(0,1)}\right|\right]$$

for $i = 1,\ldots,m-1$, since

$$0 \le \frac{1}{m}\sum_{j=1}^{i}\frac{D\left(0,\frac{j}{m}\right)}{D(0,1)}+\frac{1}{m}\sum_{j=i+1}^{m-1}\frac{D\left(\frac{j}{m},1\right)}{D(0,1)} \le 1$$

and

$$0 \le \frac{1}{n}\sum_{k=ri+1}^{r(i+1)}\frac{D\left(\frac{k}{n},\frac{i+1}{n}\right)}{D\left(\frac{i}{m},\frac{i+1}{m}\right)} \le \frac{1}{m}.$$

Moreover we have

$$0 \le \frac{1}{r}\sum_{j=1}^{k}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)\Delta_{i+1}\left(\frac{i}{m}+\frac{j}{n}\right)+\frac{1}{r}\sum_{j=k+1}^{r-1}\Delta_{i+1}\left(\frac{i}{m}+\frac{k}{n}\right)\Delta_i\left(\frac{i}{m}+\frac{j}{n}\right) \le 1.$$

Hence by (7.40) and (7.44) we get

$$(7.45)\quad \left|y_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)-y_{s+1}\left(\frac{i}{m}\right)\right| \le$$

$$\le \Delta_{i+1}\left(\frac{i}{m}+\frac{k}{n}\right)\left|D\left(\frac{i}{m},\frac{i+1}{m}\right)\right|\left[\|f'\|\cdot\|u_s\|+\left(1+\frac{1}{m}\right)\|f\|+\right.$$

$$\left.+\left|\frac{d-c}{D(0,1)}\right|\right]+\frac{1}{m}\left|D\left(\frac{i}{m},\frac{i+1}{m}\right)\right|\cdot\|f\|,$$

for $k = 1,\ldots,r-1$, $i = 1,\ldots,m-1$.

In the same way we get

$$(7.46)\quad \left|y_{s+1}\left(\frac{i}{m}-\frac{k}{n}\right)-y\left(\frac{i}{m}\right)\right| \le$$

$$\le \Delta_{i-1}\left(\frac{i}{m}-\frac{k}{n}\right)\left|D\left(\frac{i-1}{m},\frac{i}{m}\right)\right|\left[\|f'\|\cdot\|u_s\|+\left(1+\frac{1}{m}\right)\|f\|+\right.$$

$$\left.+\left|\frac{d-c}{D(0,1)}\right|\right]+\frac{1}{m}\left|D\left(\frac{i-1}{m},\frac{i}{m}\right)\right|\cdot\|f\|,$$

for $k = 1,\ldots,r-1$, $i = 1,\ldots,m-1$.

We have

$$(7.47)\quad \left|D\left(\frac{i}{m},\frac{i+1}{m}\right)\right| \le \frac{1}{m}\|1/p\|, \qquad i = 0,\ldots,m-1.$$

By (7.45) and (7.46) we get

$$(7.48)\quad |S_3| \le \frac{1}{n}\sum_{k=1-r}^{r-1}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)\|f''\|\cdot\left|y_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)-y_{s+1}\left(\frac{i}{m}\right)\|u_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)\right| \le$$

$$\le \frac{2\|1/p\|}{m^2}\left\{\|f'\|\cdot\|u_s\|+\left(1+\frac{2}{m}\right)\|f\|+\left|\frac{d-c}{D(0,1)}\right|\right\}\|f''\|\cdot\|u_{s+1}\|.$$

We will now consider S_5,

$$S_5 = \frac{1}{n}\sum_{k=1-r}^{r-1}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)f'\left(y_{s+1}\left(\frac{i}{m}\right)\right)\left[u_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)-u_{s+1}\left(\frac{i}{m}\right)\right],$$

for $i = 1,\ldots,m-1$.

By (7.15) we have

$$(7.49)\quad u_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)-u_{s+1}\left(\frac{i}{m}\right)=\Delta_{i+1}\left(\frac{i}{m}+\frac{k}{n}\right)\left[u_{s+1}\left(\frac{i+1}{m}\right)-u_{s+1}\left(\frac{i}{m}\right)\right]+$$

$$+\frac{D\left(\frac{i}{m},\frac{i+1}{m}\right)}{m}\left\{\frac{1}{r}\sum_{j=1}^{k}\Delta_i\left(\frac{i}{m}+\frac{k}{n}\right)\Delta_{i+1}\left(\frac{i}{m}+\frac{j}{n}\right)\left[f\left(y_{s+1}\left(\frac{i}{m}+\frac{j}{n}\right)\right)-f\left(y_s\left(\frac{i}{m}+\frac{j}{n}\right)\right)\right]+\right.$$

$$\left.+\frac{1}{r}\sum_{j=k+1}^{r-1}\Delta_{i+1}\left(\frac{i}{m}+\frac{k}{n}\right)\Delta_i\left(\frac{i}{m}+\frac{j}{n}\right)\left[f\left(y_{s+1}\left(\frac{i}{m}+\frac{j}{n}\right)\right)-f\left(y_s\left(\frac{i}{m}+\frac{j}{n}\right)\right)\right]\right\}=$$

$$=S_7+S_8,$$

for $k=1,\ldots,r-1$, $i=1,\ldots,m-1$, and

$$(7.50)\quad |S_8|\le\frac{\left|D\left(\frac{i}{m},\frac{i+1}{m}\right)\right|}{m}\|f'\|\,\|u_s\|.$$

We have

$$u_{s+1}\left(\frac{i+1}{m}\right)-u_s\left(\frac{i}{m}\right)=$$

$$=\frac{1}{m}\sum_{j=1}^{m-1}\left[G\left(\frac{i+1}{m},\frac{j}{m}\right)-G\left(\frac{i}{m},\frac{j}{m}\right)\right]\left[f'\left(y_{s+1}\left(\frac{j}{m}\right)\right)u_{s+1}\left(\frac{j}{m}\right)-f'\left(y_s\left(\frac{j}{m}\right)\right)u_s\left(\frac{j}{m}\right)\right]+$$

$$+\frac{1}{n}\sum_{k=1}^{n-1}\left[G\left(\frac{i+1}{m},\frac{k}{n}\right)-G\left(\frac{i}{m},\frac{k}{n}\right)\right]\left[f\left(y_{s+1}\left(\frac{k}{n}\right)\right)-f\left(y_s\left(\frac{k}{n}\right)\right)\right].$$

Hence by (7.42) and (7.43) we get

$$\left|u_{s+1}\left(\frac{i+1}{m}\right)-u_{s+1}\left(\frac{i}{m}\right)\right|\le\frac{\|1/p\|}{m}\|f'\|\left[\|u_{s+1}\|+\left(2+\frac{1}{m}\right)\|u_s\|\right],$$

and

$$\left|u_{s+1}\left(\frac{i}{m}+\frac{k}{n}\right)-u_{s+1}\left(\frac{i}{m}\right)\right|\le\frac{\|1/p\|}{m}\|f'\|\left[\|u_{s+1}\|+\left(2+\frac{2}{m}\right)\|u_s\|\right]$$

for $k=1,\ldots,r-1$, $i=1,\ldots,m-1$.

If we estimate $u_{s+1}\left(\frac{i}{m}-\frac{k}{n}\right)-u_{s+1}\left(\frac{i}{m}\right)$ in the same way we get

$$(7.51)\quad |S_5|\le\frac{2\|1/p\|}{m^2}\|f'\|^2\left\{\|u_{s+1}\|+2\left(1+\frac{1}{m}\right)\|u_s\|\right\}.$$

Now

$$S_1+S_2=S_1+S_3+S_4=S_1+S_3+S_5+S_6$$

and by (7.38), (7.48), (7.51), and (7.39) we get

$$|S_1 + S_2| \le \frac{1}{m}\|f''\|\cdot\|u_{s+1}\|^2 +$$

$$+ \frac{2\|1/p\|}{m^2}\|f'\|\cdot\|f''\|\cdot\|u_s\|\cdot\|u_{s+1}\| +$$

$$+ \left\{\frac{2\|1/p\|}{m^2}\left[(1+\frac{2}{m})\|f\|\cdot\|f''\| + \left|\frac{d-c}{D(0,1)}\right|\|f''\| + \|f'\|^2\right] +\right.$$

$$\left. + \frac{M}{6m^3}(1+\frac{1}{r^2})\|f'\|\right\}\|u_{s+1}\| + \frac{4\|1/p\|}{m^2}(1+\frac{1}{m})\|f'\|^2\|u_s\|.$$

Hence by (7.33) and (7.36) we have

$$(7.52)\quad \|u_{s+2}\|_m \le \rho\left\{\|f''\|\cdot\|u_{s+1}\|^2 +\right.$$

$$+ \frac{2\|1/p\|}{m}\|f'\|\cdot\|f''\|\cdot\|u_s\|\cdot\|u_{s+1}\| +$$

$$+ \left[\frac{2\|1/p\|}{m}\left((1+\frac{2}{m})\|f\|\cdot\|f''\| + \left|\frac{d-c}{D(0,1)}\right|\|f''\| + \|f'\|^2\right) +\right.$$

$$\left.\left. + \frac{M}{6m^2}(1+\frac{1}{r^2})\|f'\|\right]\|u_{s+1}\| + \frac{4\|1/p\|}{m}(1+\frac{1}{m})\|f'\|^2\|u_s\|\right\}.$$

By (7.15) we have

$$u_{s+2}(\tfrac{i}{m}+\tfrac{k}{n}) = \Delta_i(\tfrac{i}{m}+\tfrac{k}{n})u_{s+2}(\tfrac{i}{m}) + \Delta_{i+1}(\tfrac{i}{m}+\tfrac{k}{n})u_{s+2}(\tfrac{i+1}{m}) +$$

$$+ \frac{D(\frac{i}{m},\frac{i+1}{m})}{m}\left\{\frac{1}{r}\sum_{j=1}^{k}\Delta_i(\tfrac{i}{m}+\tfrac{k}{n})\Delta_{i+1}(\tfrac{i}{m}+\tfrac{j}{n})\left[f(y_{s+2}(\tfrac{i}{m}+\tfrac{j}{n}))-f(y_{s+1}(\tfrac{i}{m}+\tfrac{j}{n}))\right]+\right.$$

$$\left. + \frac{1}{r}\sum_{j=k+1}^{r-1}\Delta_{i+1}(\tfrac{i}{m}+\tfrac{k}{n})\Delta_i(\tfrac{i}{m}+\tfrac{j}{n})\left[f(y_{s+2}(\tfrac{i}{m}+\tfrac{j}{n})) - f(y_{s+1}(\tfrac{i}{m}+\tfrac{j}{n}))\right]\right\}$$

for $k = 1,\ldots,r-1$, $i = 0,\ldots,m-1$. Thus we have

$$(7.53)\quad \|u_{s+2}\| \le \|u_{s+2}\|_m + \frac{\|1/p\|}{m^2}\|f'\|\cdot\|u_{s+1}\|.$$

By (7.52) and (7.53) we get the following fundamental inequality

$$(7.54)\quad \|u_{s+2}\| \le \rho\Big\{\|f''\|\|u_{s+1}\|^2 +$$

$$+\frac{2\|1/p\|}{m}\|f'\|\cdot\|f''\|\cdot\|u_s\|\cdot\|u_{s+1}\| +$$

$$+\Big[\frac{2\|1/p\|}{m}\Big((1+\frac{2}{m})\|f\|\cdot\|f''\| + \Big|\frac{d-c}{D(0,1)}\Big|\cdot\|f''\| + \|f'\|^2\Big) +$$

$$+\frac{M}{6m^2}(1+\frac{1}{r^2})\|f'\|\Big]\|u_{s+1}\| + \frac{4\|1/p\|}{m}(1+\frac{1}{m})\|f'\|^2\|u_s\|\Big\} +$$

$$+\frac{\|1/p\|}{m^2}\|f'\|\cdot\|u_{s+1}\|.$$

We continue the proof by induction. We wish to prove that the statement

$$(7.55)\quad \|u_s\| \le \eta d^s \quad \text{and} \quad y_{s+1} \in S(y_0,t)$$

holds for all $s \ge 0$. For $s = 0,1$ (7.55) is valid by (7.31). Assume that (7.55) holds for $s \le N$, where $N \ge 1$. Then we have

$$\|y_{N+1} - y_0\| \le \|y_{N+1} - y_N\| + \ldots + \|y_1 - y_0\| =$$

$$= \|u_N\| + \ldots + \|u_0\| \le (d^N + \ldots + d + 1)\eta \le \frac{\eta}{1-d},$$

and $y_{N+1} \in S(y_0,t)$. For $s = N+1$ (7.54) and (7.34) give

$$\|u_{N+1}\| \le \rho\Big\{\|f''\|\cdot\|u_N\|^2 +$$

$$+\frac{2\|1/p\|}{m}\|f'\|\cdot\|f''\|\cdot\|u_N\|\cdot\|u_{N-1}\| +$$

$$+\Big[\frac{2\|1/p\|}{m}\Big((1+\frac{2}{m})\|f\|\|f''\| + \Big|\frac{d-c}{D(0,1)}\Big|\|f''\| + \|f'\|^2\Big) +$$

$$+\frac{M}{6m^2}(1+\frac{1}{r^2})\|f'\|\Big]\|u_N\| + \frac{4\|1/p\|}{m}(1+\frac{1}{m})\|f'\|^2\|u_{N-1}\|\Big\} +$$

$$+\frac{\|1/p\|}{m^2}\|f'\|\|u_N\| \le$$

$$\le \rho\Big\{\|f''\|(\eta d^N)^2 + \frac{2\|1/p\|}{m}\|f'\|\cdot\|f''\|\eta d^N\eta d^{N-1} +$$

$$+\Big[\frac{2\|1/p\|}{m}\Big((1+\frac{2}{m})\|f\|\cdot\|f''\| + \Big|\frac{d-c}{D(0,1)}\Big|\|f''\| + \|f'\|^2\Big) +$$

$$+\frac{M}{6m^2}(1+\frac{1}{r^2})\|f'\|\Big]\eta d^N + \frac{4\|1/p\|}{m}(1+\frac{1}{m})\|f'\|^2\eta d^{N-1}\Big\} +$$

$$+\frac{\|1/p\|}{m^2}\|f'\|\eta d^N \le$$

$$\le\Big\{\rho\Big\{\|f''\|\gamma+\frac{2\|1/p\|}{dm}\|f'\|\cdot\|f''\|\gamma+$$

$$+\Big[\frac{2\|1/p\|}{m}\Big((1+\frac{2}{m})\|f\|\cdot\|f''\|+\Big|\frac{d-c}{D(0,1)}\Big|\cdot\|f''\|+\|f'\|^2\Big)+$$

$$+\frac{M}{6m^2}(1+\frac{1}{r^2})\|f'\|\Big]\frac{1}{d}+\frac{4\|1/p\|}{m}(1+\frac{1}{m})\|f'\|^2\frac{1}{d^2}\Big\}+$$

$$+\frac{\|1/p\|}{m^2}\|f'\|\frac{1}{d}\Big\}\eta d^{N+1}\le\eta d^{N+1}.$$

Moreover,

$$\|y_{N+2}-y_0\|\le\|u_{N+1}\|+\ldots+\|u_0\|\le\frac{\eta}{1-d}.$$

Hence $y_{N+2}\in S(y_0,t)$, and we conclude that (7.55) is valid for all positive integers.

For $s\ge 0$, $\nu\ge 1$, we have

$$(7.56)\quad \|y_{s+\nu}-y_s\|\le\|u_{s+\nu-1}\|+\ldots+\|u_s\|\le$$

$$\le(d^{s+\nu-1}+\ldots+d^s)\eta=(d^{\nu-1}+\ldots+1)d^s\eta\le\frac{d^s\eta}{1-d}$$

which shows that the sequence $\{y_s\}$ is a Cauchy sequence and hence, because $\bar{S}(y_0,t)$ is closed and bounded, it has a limit $y^*\in\bar{S}(y_0,t)$. Consider the Newton-like method, which in matrix notation can be written as

$$(7.57)\quad [T_n-\frac{1}{m}D(f'(y_s))]u_s=-[T_ny_s-y_{01}-\frac{1}{n}f(y_s)]$$

where D is defined by (7.28). Hence we have

$$(7.58)\quad \|T_ny_s-y_{01}-\tfrac{1}{n}f(y_s)\|\le\|T_n-\tfrac{1}{m}D(f'(y_s))\|\cdot\|u_s\|,$$

and since $\|u_s\|\to 0$ as $s\to\infty$ and the $(n-1)\times(n-1)$ matrix $T_n-\frac{1}{m}D(f'(y_s))$ is uniformly bounded on $S(y_0,t)$ the right hand side of (7.58) tends to zero as $s\to\infty$. Hence

$$T_ny_s-y_{01}-\tfrac{1}{n}f(y_s)\to 0\quad\text{as}\quad s\to\infty$$

and the continuity of $T_n y - y_{01} - \frac{1}{n} f(y)$ implies that $T_n y^* - y_{01} - \frac{1}{m} f(y^*) = 0$. Hence y^* satisfies (7.12).

Finally, the estimate (7.35) follows from the inequality (7.56) by letting ν tends to $+\infty$. The solution y^* is unique in $\overline{S}(y_0,t)$ since (7.57) defines a contraction there.

Remark.

If $f'(y) \geq 0$ for all y then we have

$$- T_m + D(f'(y)) \leq - T_m$$

since $-T_m$ is an M-matrix. We have

$$\| T_m \| = m \max_{1 \leq i \leq m-1} \left\{ \frac{1}{m} \sum_{j=1}^{m-1} -G\left(\frac{i}{m},\frac{j}{m}\right) \right\}.$$

Hence in this case we have

$$\beta = \max_{1 \leq i \leq m-1} \left\{ \frac{1}{m} \sum_{j=1}^{m-1} -G\left(\frac{i}{m},\frac{j}{m}\right) \right\}.$$

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8. EXTENSIONS

We have in this report considered Newton-like methods for the numerical solution of integral equations of the form

$$(8.1) \quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + h(x), \quad 0 \le x \le 1,$$

where

$$(8.2) \quad G(x,t) = \begin{cases} b(x)a(t), & x \ge t, \\ a(x)b(t), & x \le t. \end{cases}$$

The kernel $G(x,t)$ is called <u>semidegenerate</u>. Generally a kernel K is called semidegenerate if it has the form

$$(8.3) \quad K(x,t) = \begin{cases} \sum_{i=1}^{M} a_i(x)b_i(t), & x > t, \\ \sum_{j=1}^{N} c_j(x)d_j(t), & x < t. \end{cases}$$

Newton-like methods for the numerical solution of integral equations of the form

$$(8.4) \quad y(x) = \int_0^1 K(x,t)f(t,y(t))dt + h(x), \quad 0 \le x \le 1,$$

have been given in Aulin [7]. These methods can also be extended to nonlinear equations of Urysohn type

$$(8.5) \quad y(x) = \int_0^1 K(x,t,y(t))dt + h(x), \quad 0 \le x \le 1,$$

where

$$K(x,t,y(t)) = \begin{cases} \sum_{i=1}^{M} a_i(x)b_i(t,y(t)), & x > t, \\ \sum_{j=1}^{N} c_j(x)d_j(t,y(t)), & x < t. \end{cases}$$

In this report we have only considered symmetric boundary value problems. For continuous problems the symmetric and the unsymmetric, $Ly = p_0y'' + p_1y' + p_2y$ and $p_0' \neq p_1$, cases are identical if Green's function is known since the Wronskian is

$$ab' - a'b = C\exp\left[-\int(p_1/p_0)dt\right],$$

where a and b are two linearly independent solutions of the homogeneous equation $Ly = 0$, and $\exp\left[\int(p_1/p_0)dt\right]L$ is symmetric. The determinant formulation of the unsymmetric discrete case has been treated in Aulin [10] and Newton-like methods for these equations can be derived exactly as in chapter 6.

In Aulin [8] Newton-like methods for the numerical solution of the boundary value problem $u_{xx} + u_{yy} = f(x,y,u(x,y))$, u given on the boundary of the unit square, are given.

9. NUMERICAL EXAMPLES

In this chapter let

$$\|g\| = \max_{0 \le k \le n} \left|g(\tfrac{k}{n})\right|$$

and let $u_s(x) = y_{s+1}(x) - y_s(x)$ be the correction of the sth iteration. Furthermore we will consider combinations of the grids

$$G_n = \left\{ \frac{k}{n} \;\middle|\; k = 0,\ldots,n \right\},$$

where $n = 2^i$, $i = 0,1,\ldots,9$. The number of points in the grids G_m and G_n of Newton-like methods will be denoted by $(m+1)$-$(n+1)$, where $m \le n$. Let $G_0 = \emptyset$, the empty set.

For Newton's method the number of points in the grid will be denoted by $(n+1)$-$(n+1)$, and for the method of successive approximations the number of points in the grid will be denoted by 0-$(n+1)$.

The exact solutions of the boundary value problems considered in this chapter will be denoted by $y(x)$ and the error of the computed solution y_s will be denoted by $\|y_s - y\|$.

Example 9.1

Consider the boundary value problem

$$(9.1)\quad \begin{cases} (\cos^2(t)y')' = [y^3\cos^2(t) + y^2\sin(2t)]/2, & 0 < t < 1, \\ 2y(0) - 2y'(0) = 3, \\ 9y(1) + 9y'(1) = 4. \end{cases}$$

The exact solution of this problem is

$$(9.2)\quad y(t) = \frac{1}{1 + \frac{t}{2}}, \qquad 0 \le t \le 1.$$

We approximate (9.1) by (4.11) on the grid G_{512}. For this problem we have

$$(9.3) \quad p(t) = \cos^2(t), \quad 0 \le t \le 1,$$

and

$$(9.4) \quad D(x,x+z) = \frac{-\sin(z)}{\cos(x)\cos(x+z)}, \quad 0 \le x \le x+z \le 1.$$

We have solved this problem by the Newton-like methods defined by (4.14) and (4.15) and as a first approximation we have used

$$(9.5) \quad y_0(t) = \frac{62 - 19t}{54}, \quad 0 \le t \le 1.$$

The results are shown in table 9.1.

We have also solved this problem when D was approximated by the midpoint rule

$$D(\tfrac{i}{512},\tfrac{i+1}{512}) := \frac{-1}{512}\, p(\tfrac{i+\frac{1}{2}}{512}), \quad i = 0,\ldots,511.$$

We got the same results as in table 9.1 except for the errors which were

on G_{64}: $\| y_8 - y \| = 1.49E\text{-}05$,

G_{128}: $\| y_8 - y \| = 3.72E\text{-}06$,

G_{256}: $\| y_8 - y \| = 9.30E\text{-}06$,

G_{512}: $\| y_8 - y \| = 2.32E\text{-}07$.

Table 9.1

Number of points in the grids	$\|\|u_s\|\|$ of iteration number								$\|\|y_8-y\|\|$	Computer time per iteration (in ms)
	0	1	2	3	4	5	6	7		
0-513	4.2E-01	4.9E-01	6.2E-01	7.0E-01	8.5E-01	9.1E-01	1.0E+00	1.1E-01	4.16E-01	600
2-513	2.4E-01	9.8E-02	4.7E-02	2.1E-02	9.9E-03	4.5E-03	2.1E-03	9.5E-04	3.00E-04	600
3-513	1.8E-01	9.6E-03	5.0E-03	5.2E-04	1.4E-04	2.0E-05	4.4E-06	7.1E-07	2.14E-07	600
5-513	1.6E-01	1.2E-02	1.1E-03	5.2E-05	5.8E-06	2.1E-07	2.9E-08	7.7E-10	1.13E-07	600
9-513	1.6E-01	1.3E-02	3.5E-04	1.8E-05	4.3E-07	2.4E-08	5.1E-10	3.1E-11	1.13E-07	600
17-513	1.6E-01	1.4E-02	1.6E-04	4.8E-06	5.2E-08	1.6E-09	1.6E-11	5.4E-13	1.13E-07	600
33-513	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.6E-09	1.0E-10	8.0E-13	8.7E-15	1.13E-07	650
65-513	1.6E-01	1.4E-02	1.0E-04	3.1E-07	2.2E-09	6.4E-12	4.6E-14	1.3E-16	1.13E-07	700
129-513	1.6E-01	1.4E-02	9.8E-05	7.7E-08	5.0E-10	3.8E-13	2.5E-15	3.5E-18	1.13E-07	900
257-513	1.6E-01	1.4E-02	9.7E-05	1.9E-08	1.0E-10	1.9E-14	1.0E-16	1.7E-18	1.13E-07	1 150
513-513	1.6E-01	1.4E-02	9.7E-05	4.7E-09	1.2E-11	2.2E-18	8.3E-19	8.3E-19	1.13E-07	1 400
33-65	1.6E-01	1.4E-02	1.1E-04	9.2E-07	6.9E-09	5.8E-11	4.3E-13	3.6E-15	7.21E-06	100
33-129	1.6E-01	1.4E-02	1.1E-04	1.2E-06	8.9E-09	9.2E-11	7.0E-13	7.2E-15	1.80E-06	200
33-257	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.5E-09	1.0E-10	7.8E-13	8.4E-15	4.51E-07	350

Example 9.2

Consider the boundary value problem (9.1). We approximate this problem by the system defined by (4.26), (4.27), and (4.28). We have solved this system by the Newton-like methods defined by (3.25), (3.26), and (3.27) followed by (3.29). As a first approximation we have throughout this example used y_0 in (9.5). The results are shown in table 9.2. After that we have solved the system (4.26), (4.27), and (4.28) by the Newton-like methods defined by (4.22), (4.23), and (4.24) followed by (4.25). The results are shown in table 9.3. We can see that the rate of convergence of these two methods are almost the same.

We have also solved this problem by the Newton-like methods defined by (4.22), (4.23), (4.24), (4.25), where D was approximated by Simpson's rule

$$D\left(\frac{i}{512},\frac{i+1}{512}\right) := \frac{-1}{3072}\left[\frac{1}{p\left(\frac{i}{512}\right)} + \frac{4}{p\left(\frac{i+\frac{1}{2}}{512}\right)} + \frac{1}{p\left(\frac{i+1}{512}\right)}\right], \quad i = 0,\ldots,511.$$

We got the same results as in table 9.3 except for the errors which were

$$\begin{aligned} \text{on } G_{64}&: \|y_8 - y\| = 1.31\text{E-}09,\\ G_{128}&: \|y_8 - y\| = 8.35\text{E-}11,\\ G_{256}&: \|y_8 - y\| = 5.28\text{E-}12,\\ G_{512}&: \|y_8 - y\| = 3.32\text{E-}13. \end{aligned}$$

Example 9.3

In this example we will solve the boundary value problem (9.1) by usinf the $O(h^4)$ approximation defined by (4.26), (4.27), (4.28). To solve this system we will use Newton's method where f'_y only is evaluated on the grid G_m and f'_y is defined on $G_n \setminus G_m$ $(m \quad n)$ by linear interpolation. As a first approximation we have used y_0 in (9.5). The results are shown in table 9.4.

Table 9.2

Number of points in the grids	$\|\| u_s \|\|$ of iteration number								$\|\| y_8 - y \|\|$	Computer time per iteration (in ms)
	0	1	2	3	4	5	6	7		
3-513	1.7E-01	9.5E-03	4.6E-03	2.3E-04	1.1E-04	8.9E-06	2.8E-06	3.0E-07	6.60E-08	700
5-513	1.6E-01	1.3E-02	1.2E-03	6.5E-05	6.3E-06	3.1E-07	3.3E-08	1.5E-09	1.77E-10	650
9-513	1.6E-01	1.4E-02	3.7E-04	1.9E-05	4.9E-07	2.5E-08	6.2E-10	3.4E-11	9.82E-13	700
17-513	1.6E-01	1.4E-02	1.7E-04	4.9E-06	5.6E-08	1.7E-09	1.8E-11	5.7E-13	3.09E-13	650
33-513	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.8E-09	1.1E-10	8.3E-13	8.9E-15	3.05E-13	700
65-513	1.6E-01	1.4E-02	1.0E-04	3.1E-07	2.2E-09	6.5E-12	4.6E-14	1.4E-16	3.05E-13	750
129-513	1.6E-01	1.4E-02	9.8E-05	7.7E-08	5.0E-10	3.9E-13	2.5E-15	3.5E-18	3.05E-13	850
257-513	1.6E-01	1.4E-02	9.7E-05	1.9E-08	1.0E-10	1.9E-14	1.0E-16	8.7E-19	3.05E-13	1 100
513-513	1.6E-01	1.4E-02	9.7E-05	4.7E-09	1.1E-17	7.7E-19	7.7E-19	7.7E-19	3.05E-13	1 500
33- 65	1.6E-01	1.4E-02	1.1E-04	9.4E-07	7.2E-09	6.0E-11	4.6E-13	3.9E-15	1.20E-09	140
33-129	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.2E-09	9.4E-11	7.3E-13	7.5E-15	7.67E-11	220
33-257	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.7E-09	1.0E-10	8.1E-13	8.6E-15	4.85E-12	370

Table 9.3

Number of points in the grids	$\|\|u_s\|\|$ of iteration number								$\|\|y_8-y\|\|$	Computer time per iteration (in ms)
	0	1	2	3	4	5	6	7		
0-513	4.2E-01	5.0E-01	6.2E-01	7.0E-01	8.5E-01	9.1E-01	1.0E+00	1.1E+00	4.16E-01	600
2-513	2.4E-01	9.8E-02	4.7E-02	2.1E-02	9.9E-03	4.5E-03	2.1E-03	9.5E-04	3.03E-04	650
3-513	1.8E-01	9.6E-03	5.0E-03	5.2E-04	1.4E-04	2.0E-05	4.4E-06	7.1E-07	1.20E-07	650
5-513	1.6E-01	1.2E-02	1.1E-03	5.2E-05	5.8E-06	2.1E-07	2.9E-08	7.7E-10	1.52E-10	650
9-513	1.6E-01	1.3E-02	3.5E-04	1.8E-05	4.3E-07	2.4E-08	5.1E-10	3.1E-11	7.83E-13	650
17-513	1.6E-01	1.4E-02	1.6E-04	4.8E-06	5.2E-08	1.6E-09	1.6E-11	5.4E-13	3.08E-13	650
33-513	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.6E-09	1.0E-10	8.0E-13	8.7E-15	3.05E-13	700
65-513	1.6E-01	1.4E-02	1.0E-04	3.1E-07	2.2E-09	6.4E-12	4.6E-14	1.3E-16	3.05E-13	750
129-513	1.6E-01	1.4E-02	9.8E-04	7.7E-08	5.0E-10	3.8E-13	2.5E-15	4.3E-18	3.05E-13	850
257-513	1.6E-01	1.4E-02	9.7E-05	1.9E-08	1.0E-10	1.9E-14	1.0E-16	1.7E-18	3.05E-13	1 000
513-513	1.6E-01	1.4E-02	9.7E-05	4.7E-09	2.5E-15	1.6E-18	1.6E-18	1.6E-18	3.05E-13	1 350
33- 65	1.6E-01	1.4E-02	1.1E-04	9.3E-07	7.0E-09	5.9E-11	4.4E-13	3.7E-15	1.20E-09	140
33-129	1.6E-01	1.4E-02	1.1E-04	1.2E-06	8.9E-09	9.2E-11	7.0E-13	7.3E-15	7.67E-11	210
33-257	1.6E-01	1.4E-02	1.1E-04	1.2E-06	9.5E-09	1.0E-10	7.8E-13	8.4E-15	4.85E-12	360

Table 9.4

Number of points in the grids	$\|\|u_s\|\|$ of iteration number							$\|\|y_s-y\|\|$	Computer time per iteration (in ms)
	0	1	2	3	4	5	6	7		
2-513	1.8E-01	5.3E-03	4.2E-04	3.5E-05	2.9E-06	2.4E-07	2.0E-08	1.6E-09	1.05E-10	1 000
3-513	1.6E-01	1.0E-02	1.3E-04	2.3E-06	4.1E-08	7.4E-10	1.3E-11	2.3E-13	3.02E-13	1 050
5-513	1.6E-01	1.3E-02	2.9E-05	1.1E-07	4.7E-10	2.0E-12	8.9E-15	3.8E-17	3.05E-13	1 050
9-513	1.6E-01	1.4E-02	7.9E-05	8.0E-08	8.5E-11	9.2E-14	9.8E-17	8.3E-19	3.05E-13	1 050
17-513	1.6E-01	1.4E-02	9.3E-05	2.0E-08	5.4E-12	1.4E-15	1.6E-18	1.6E-18	3.05E-13	1 050
33-513	1.6E-01	1.4E-02	9.6E-05	1.9E-09	1.2E-13	8.5E-18	1.7E-18	1.7E-18	3.05E-13	1 050
65-513	1.6E-01	1.4E-02	9.7E-05	3.1E-09	4.9E-14	8.6E-19	8.6E-19	8.6E-19	3.05E-13	1 100
129-513	1.6E-01	1.4E-02	9.7E-05	4.3E-09	1.6E-14	1.7E-18	1.7E-18	1.7E-18	3.05E-13	1 150
257-513	1.6E-01	1.4E-02	9.7E-05	4.6E-09	3.5E-15	1.3E-18	1.7E-18	1.7E-18	3.05E-13	1 250

Example 9.4

The approximation (4.11) of the boundary value problem (9.1) defines a discrete tridiagonal system for the unknowns $y(0),\ldots,y(1)$. In this example we have solved this system by the Newton-like methods given by (6.19), (6.20), and (6.21) followed by (6.22). To compute $\Delta_{i,j}$ we have used the algorithm given on p. 60. Furthermore the matrix of the tridiagonal system (6.19), (6.20), and (6.21) has been evaluated according to Theorem 6.3. As a first approximation we have throughout this example used y_0 in (9.5). The results are shown in table 9.5.

Consider the linear boundary value problem

$$(9.6)\quad \begin{cases} (\cos^2(t)y')' = \left[\dfrac{\cos^2(t)}{(1+\frac{t}{2})^3} + \dfrac{\sin(2t)}{(1+\frac{t}{2})^2}\right]/2, & 0 < t < 1, \\ 2y(0) - 2y'(0) = 3, \\ 9y(1) + 9y'(1) = 4, \end{cases}$$

i.e. problem (9.1) where y of the right hand side has been replaced by $1/(1 + t/2)$. Consider the $O(h^4)$ approximation, defined by the system (4.26), (4.27), (4.28), of this problem. We have solved this system by the Newton-like methods defined by (6.19), (6.20), (6.21), and (6.22). (We have $f'_y = 0$.) The results are shown in table 9.6. For comparison we have also solved this system by the Newton-like methods defined by (4.22), (4.23), (4.24), and (4.25). The results are shown in table 9.7.

In table 9.6 we see how the error of the vectors $\Delta_{i,j}$ influence the errors of the computed solutions of the Newton-like methods 2-513, 3-513, 5-513, 9-513, 17-513. The influence of the condition numbers of the matrices of the first step in the other Newton-like methods are reflected in the second correction u_1. In table 9.7 we have used the exact values of $\Delta_{i,j}$.

Table 9.5

Number of points in the grids	$\|\|\bar{u}_s\|\|$ of iteration number								$\|\|y_8-y\|\|$	Computer time per iteration (in ms)
	0	1	2	3	4	5	6	7		
2-513	2.4E-01	9.8E-02	4.7E-02	2.1E-02	9.8E-03	4.5E-03	2.1E-03	9.5E-04	2.99E-04	500
3-513	1.9E-01	3.3E-02	1.1E-02	2.6E-03	7.1E-04	1.8E-04	4.7E-05	1.2E-05	2.65E-06	550
5-513	1.7E-01	4.7E-03	1.3E-03	8.7E-05	1.3E-05	1.4E-06	1.7E-07	1.9E-08	1.14E-07	550
9-513	1.6E-01	1.0E-02	1.8E-04	1.2E-05	1.7E-07	2.0E-08	7.9E-10	5.2E-11	1.13E-07	550
17-513	1.6E-01	1.2E-02	3.6E-05	4.6E-06	6.5E-08	2.5E-09	5.4E-11	1.6E-12	1.13E-07	600
33-513	1.6E-01	1.3E-02	4.7E-05	1.1E-06	4.3E-09	1.2E-10	1.0E-12	1.7E-14	1.13E-07	650
65-513	1.6E-01	1.4E-02	7.0E-05	1.9E-07	1.2E-09	1.8E-12	2.3E-14	2.8E-16	1.13E-07	700
129-513	1.6E-01	1.4E-02	8.6E-05	1.8E-08	4.5E-10	4.9E-13	2.5E-15	1.0E-15	1.13E-07	800
257-513	1.6E-01	1.4E-02	9.5E-05	5.0E-09	9.8E-11	1.8E-14	3.6E-16	2.0E-15	1.13E-07	1 050
33- 65	1.6E-01	1.4E-02	9.3E-05	8.4E-07	5.1E-09	4.9E-11	2.7E-13	3.0E-15	7.21E-06	140
33-129	1.6E-01	1.3E-02	6.7E-05	9.9E-07	2.1E-09	7.9E-11	2.5E-13	7.2E-15	1.90E-06	210
33-257	1.6E-01	1.3E-02	5.3E-05	1.1E-06	2.7E-09	1.0E-10	7.3E-13	1.2E-14	4.51E-07	380

Table 9.6

Number of points in the grids	$\|\|u_s\|\|$ of iteration number				$\|\|y_3-y\|\|$
	0	1	2	3	
2-513	1.7E-01	1.1E-18	1.1E-18	1.1E-18	5.1632E-13
3-513	1.7E-01	2.6E-18	5.3E-18	5.3E-18	5.0784E-13
5-513	1.7E-01	2.6E-18	8.3E-19	8.3E-19	5.0558E-13
9-513	1.7E-01	1.0E-17	8.7E-19	8.7E-19	5.0512E-13
17-513	1.7E-01	3.4E-17	9.3E-19	9.3E-19	5.0500E-13
33-513	1.7E-01	1.2E-16	8.4E-19	8.4E-19	5.0496E-13
65-513	1.7E-01	5.0E-16	1.7E-18	1.7E-18	5.0495E-13
129-513	1.7E-01	1.6E-15	1.4E-18	1.4E-18	5.0495E-13
257-513	1.7E-01	8.3E-15	8.8E-19	8.8E-19	5.0495E-13
513-513	1.7E-01	3.2E-14	1.6E-18	1.3E-18	5.0495E-13

Table 9.7

Number of points in the grids	$\|\|u_s\|\|$ of iteration number				$\|\|y_3-y\|\|$
	0	1	2	3	
3-513	1.7E-01	3.5E-18	1.0E-18	1.0E-18	5.0496E-13
5-513	1.7E-01	2.8E-18	1.3E-18	1.3E-18	5.0495E-13
9-513	1.7E-01	8.7E-18	1.7E-18	9.2E-19	5.0495E-13
17-513	1.7E-01	4.6E-17	1.2E-18	1.2E-18	5.0495E-13
33-513	1.7E-01	1.0E-16	2.0E-18	4.8E-19	5.0495E-13
65-513	1.7E-01	3.8E-16	8.5E-19	8.5E-19	5.0495E-13
129-513	1.7E-01	1.9E-15	1.7E-18	8.6E-19	5.0495E-13
257-513	1.7E-01	8.1E-15	1.7E-18	8.7E-19	5.0495E-13
513-513	1.7E-01	3.2E-14	7.3E-19	7.3E-19	5.0495E-13

Remark.

In equations (6.19), (6.20), and (6.21) we have used the following expressions

$$\frac{a_{n_1}}{a_0 d(0,n_1)} y_0 - y_{n_1}/d(0,n_1) = \frac{1}{d(0,n_1)} [y_0 - y_{n_1}] +$$

$$+ \frac{p(0)}{a_{12}} [c - a_{11} y_0],$$

$$\frac{- y_{n_{i-1}}}{d(n_{i-1},n_i)} + \frac{d(n_{i-1},n_{i+1})}{d(n_{i-1},n_i) d(n_i,n_{i+1})} y_{n_i} + \frac{- y_{n_{i+1}}}{d(n_i,n_{i+1})} =$$

$$= \frac{1}{d(n_{i-1},n_i)} [y_{n_i} - y_{n_{i-1}}] + \frac{1}{d(n_i,n_{i+1})} [y_{n_i} - y_{n_{i+1}}],$$

and

$$\frac{- y_{n_{m-1}}}{d(n_{m-1},n)} + \frac{b_{n_{m-1}}}{b_n d(n_{m-1},n)} = \frac{1}{d(n_{m-1},n)} [y_n - y_{n_{m-1}}] +$$

$$+ \frac{p(1)}{b_{22}} [d - b_{21} y_n],$$

for the linear boundary value problem. If the left hand sides of this expressions are not evaluated according to the right hand sides the example with the linear equations does not make sense. (See also Pohl [15].)

For the numerical solution of the linear tridiagonal systems in this chapter we have used the procedures "banddet1" and "bandsol1" in Wilkinson and Reinsch [17].

All the computations were carried out at the Lund University Computing Center on a UNIVAC 1108 using ALGOL and double precision arithmetic (60 bits in the mantissa).

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